Vol. 71, No. 5, December 1998

MATHEMATICS MAGAZINE

- Galimatias Arithmeticae
- The Newest Inductee in the Number Hall of Fame
- The Crossing Number of $C_m \times C_n$
- Leibniz's Formula, Cauchy Majorants, and Linear Differential Equations

An Official Publication of The MATHEMATICAL ASSOCIATION OF AMERICA

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ARTICLES

Galimatias Arithmeticae

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You may read in the *Oxford English Dictionary* that *galimatias* means confused language, meaningless talk. This is what you must expect in this talk.[1] As a token of admiration to Gauss, I dare to append the word *Arithmeticae* to my title. I mean no offense to the Prince, who, at age 24, published *Disquisitiones Arithmeticae*, the imperishable masterwork.

As I retire (or am hit by retirement), it is time to look back at events in my career. Unlike what most people do, I would rather talk about mathematical properties and problems of some numbers connected with highlights of my life. I leave for the end the most striking conjunction.

I'll begin with the hopeful number 11 and end with the ominous number 65.

11

- At age 11 I learned how to use x to represent an unknown quantity in order to solve problems like this one: "Three brothers, born two years apart, had sums of ages equal to 33. What are their ages?" The power of the method was immediately clear to me and determined that I would be interested in numbers, even after my age would surpass the double of the sum of the ages of the three brothers.

  But 11 is interesting for many better reasons.

- 11 is the smallest prime repunit. A number with n digits all equal to 1 is called a *repunit* and denoted by R_n. So $11 = R_2$. The following repunits are known to be prime: R_n with $n = 2$, 19, 23, 317, and 1031. It is not known whether there are infinitely many prime repunits.
- If $n > 11$, there exists a prime $p > 11$ such that

$$p \text{ divides } n(n+1)(n+2)(n+3).$$

  A curiosity? Not quite. A good theorem (by Mahler) states that if $f(x)$ is a polynomial with integral coefficients of degree two or more (for two, the theorem is Pólya's), and if H is a finite set of primes (such as $\{2,3,5,7,11\}$) then there exists n_0 such that if all prime factors of $f(n)$ are in H, then $n \le n_0$.

  Another way of expressing this fact is as follows: $\lim_{n\to\infty} P[f(n)] = \infty$, where $P[f(n)]$ denotes the largest prime factor of $f(n)$. With the theory of Baker on linear forms in logarithms, Coates gave an effective bound for n_0. For the particular polynomial $f(x) = x(x+1)(x+2)(x+3)$, the proof is elementary.

[1] This paper is a modified version of a talk at the University of Munich, given in November 1994 at a festive colloquium in honor of Professor Sibylla Priess-Crampe.

- 11 is the largest positive integer d that is square-free and such that $\mathbb{Q}(\sqrt{-d})$ has a euclidean ring of integers. The other such fields are those with $d = 1$, 2, 3, and 7. This means that if $\alpha, \beta \in \mathbb{Z}[\sqrt{-d}\,]$, there exist $\gamma, \delta \in \mathbb{Z}[\sqrt{-d}\,]$ such that $\alpha = \beta\gamma + \delta$ where $\delta = 0$ or $N(\delta) < N(\beta)$. (Here, for $\alpha = a + b\sqrt{-d}$, $N(\alpha) = a^2 + db^2$. The situation is just like that for euclidean division in the ring $\mathbb{Z}$ of ordinary integers.)
- It is not known whether there exists a cuboid with sides a, b, and c measured in integers, as well as all diagonals measured in integers. In other words, it is not known whether the following system has a solution in non-zero integers:

$$\begin{cases} a^2 + b^2 = d^2 \\ b^2 + c^2 = e^2 \\ c^2 + a^2 = f^2 \\ a^2 + b^2 + c^2 = g^2 \end{cases}$$

  If such integers exist, then 11 divides abc.
- 11 is the smallest integer that is not a *numerus idoneus*.

  You don't know what a *numerus idoneus* is? I too needed to reach 65 before realizing how this age and *idoneus* numbers are connected with each other. So be patient.
- According to the theory of supersymmetry, the world has 11 dimensions: 3 for space position, 1 for time, and 7 to describe the various possible superstrings and their different vibrating patterns, so explaining subatomic particles' behavior.

  Is this a joke or a new theory to explain the world?
- The Mersenne numbers are the integers $M_q = 2^q - 1$, where q is a prime. Big deal: some are prime, some are composite. Bigger deal: how many of each kind? Total mystery!

  $M_{11} = 2^{11} - 1 = 2047 = 23 \cdot 28$. It is the smallest composite Mersenne number. The largest known composite Mersenne number is M_q, with

$$q = 8069496435 \times 10^{5072} - 1.$$

19

- One of my favorite numbers has always been 19. At this age Napoleon was winning battles—this we should forget. At the same age, Gauss discovered the law of quadratic reciprocity—this you cannot forget, once you have known it.
- First a curiosity concerning the number 19. It is the largest integer n such that

$$n! - (n-1)! + (n-2)! - \cdots \pm 1!$$

  is a prime number. The other integers n with this property are

$$n = 3, 4, 5, 6, 7, 8, 9, 10, \quad \text{and} \quad 15.$$

- Both the repunit R_{19} and the Mersenne number M_{19} are prime numbers.
- Let $U_0 = 0$, $U_1 = 1$, and $U_n = U_{n-1} + U_{n-2}$ for $n \geq 2$; these are the Fibonacci numbers. If U_n is prime, then n must also be prime, but not conversely. 19 is the smallest prime index that provides a counterexample: $U_{19} = 4181 = 37 \cdot 113$.
- The fields $\mathbb{Q}(\sqrt{-19})$, $\mathbb{Q}(\sqrt{19})$ have class number 1. (The class number is a natural number which one associates to every number field. It is 1 for the field of rationals; it is also 1 for the field of Gaussian numbers, and for any field whose arithmetical

properties resemble those of the rational numbers. The larger the class number of a number field, the more its arithmetical properties "deviate" from those of the rationals. For more on these concepts, see [3].) The ring of integers of $\mathbb{Q}(\sqrt{19})$ is euclidean, while the ring of integers of $\mathbb{Q}(\sqrt{-19})$ is not euclidean.

- Let $n > 2$, $n \not\equiv 2 \pmod 4$, and let $\zeta_n = e^{2\pi i/n}$ denote a primitive n-th root of 1. 19 is the largest prime p such that $\mathbb{Q}(\zeta_p)$ has class number 1. This was important in connection with Kummer's research on Fermat's last theorem.

  Masley and Montgomery determined in 1976 all integers n, $n \not\equiv 2 \pmod 4$, such that $\mathbb{Q}(\zeta_n)$ has class number 1, namely:

$$n = 1,3,4,5,7,8,9,11,12,13,15,16,17,19,20,21,24,25,27,28,\\ 32,33,35,36,40,44,45,48,60, \text{ and } 84.$$

- Balasubramanian, Dress, and Deshouillers showed in 1986 that every natural number is the sum of at most 19 fourth powers. Davenport had shown in 1939 that every sufficiently large natural number is the sum of at most 16 fourth powers. This provided a complete solution of the two forms of Waring's problem for fourth powers.

29

- Twin primes, such as 29 and 31, are not like the ages of twins—their difference is 2. Why? There are many twin persons and many twin primes, but in both cases, it is not known whether there are infinitely many...

  Euler showed that

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty.$$

  On the other hand, Brun showed that

$$\sum_{p,\, p+2 \text{ primes}} \frac{1}{p} < \infty.$$

  Brun's result says that either there are only finitely many twin primes, or, if there are infinitely many twin primes, their size must increase so rapidly that the sum above remains bounded. All of this is amply discussed in my book on prime numbers [5].

- A curiosity observed by Euler: If 29 divides the sum $a^4 + b^4 + c^4$, then 29 divides $\gcd(a, b, c)$.
- Let p be a prime. The *primorial* of p is

$$p\# = \prod_{q \le p,\, q \text{ prime}} q;$$

  $29 = 5\# - 1$. The expressions $p\# + 1$ and $p\# - 1$ have been considered in connection with variants of Euclid's proof that there exist infinitely many primes. The following primes p are the only ones less than or equal to 11213 such that $p\# - 1$ is prime:

$$p = 3,5,11,13,41,89,317,991,1873,2053.$$

  For this and similar sequences, see [5].

- $2 \cdot 29^2 - 1 = \square$ (a square); similarly $2 \cdot 1^2 - 1 = \square$, $2 \cdot 5^2 - 1 = \square$. In fact, there are infinitely many natural numbers x such that $2x^2 - 1 = \square$. Here is how to obtain all pairs of natural numbers (t, x) such that $t^2 - 2x^2 = -1$. From $(t + \sqrt{2}x)(t - \sqrt{2}x) = -1$, it follows that $t + \sqrt{2}x$ is a unit in the field $\mathbb{Q}(\sqrt{2})$. The fundamental unit is $1 + \sqrt{2}$ with the norm $(1 + \sqrt{2})(1 - \sqrt{2}) = -1$, so $t + \sqrt{2}x = (1 + \sqrt{2})^n$ with n odd. Thus we have

$$(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}; \quad (1 + \sqrt{2})^3 = 7 + 5\sqrt{2}; \quad (1 + \sqrt{2})^5 = 41 + 29\sqrt{2}.$$

The next solution is obtained from

$$(1 + \sqrt{2})^7 = 239 + 169\sqrt{2},$$

namely, $2 \cdot 169^2 - 1 = 239^2$.

- The ring of integers of $\mathbb{Q}(\sqrt{29})$ is euclidean. There are 16 real quadratic fields $\mathbb{Q}(\sqrt{d})$ with a euclidean ring of integers, namely

$$d = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73.$$

- $2X^2 + 29$ is an *optimal prime-producing polynomial*. Such polynomials were first considered by Euler—they are polynomials $f \in \mathbb{Z}[X]$ that assume as many initial prime values as they possibly can. More precisely, let $f \in \mathbb{Z}[X]$, with positive leading coefficient and $f(0) = q$, a prime. There exists the smallest $r > 0$ such that $f(r) > q$ and $q \mid f(r)$. The polynomial is *optimal prime-producing* if $f(k)$ is prime for $k = 0, 1, \ldots, r - 1$.

  Euler observed that $X^2 + X + 41$ is optimal prime-producing, since it assumes prime values at $k = 0, 1, \ldots, 39$, while $40^2 + 40 + 41 = 41^2$.

  In 1912, Rabinovitch showed that the polynomial $f(X) = X^2 + X + q$ (with q prime) is optimal prime-producing if and only if the field $\mathbb{Q}(\sqrt{1 - 4q})$ has class number 1.

  Heegner, Stark, and Baker determined all the imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ (with $d < 0$ and d square-free) with class number 1:

$$d = -1, -2, -5, -7, -11, -19, -43, -67, -163.$$

  These correspond to the only optimal prime-producing polynomials of the form $X^2 + X + q$, namely $q = 2, 3, 5, 11, 17, 41$. $X^2 + X + 41$ is the record prime-producing polynomial of the form $X^2 + X + q$.

  Frobenius (1912) and Hendy (1974) studied optimal prime-producing polynomials in relation to imaginary quadratic fields having class number 2. There are three types of such fields:

  (i) $\mathbb{Q}(\sqrt{-2p})$, where p is an odd prime;
  (ii) $\mathbb{Q}(\sqrt{-p})$, where p is prime and $p \equiv 1 \pmod 4$;
  (iii) $\mathbb{Q}(\sqrt{-pq})$ where p, q are odd primes, with $p < q$ and $pq \equiv 3 \pmod 4$.

  For the types of fields above, the following theorem holds:

  (i) $\mathbb{Q}(\sqrt{-2p})$ has class number 2 if and only if $2X^2 + p$ assumes prime values at $k = 0, 1, \ldots, p - 1$.
  (ii) $\mathbb{Q}(\sqrt{-p})$ has class number 2 if and only if $2X^2 + 2X + \frac{p+1}{2}$ assumes prime values at $k = 0, 1, \ldots, \frac{p-3}{2}$.
  (iii) $\mathbb{Q}(\sqrt{-pq})$ has class number 2 if and only if $pX^2 + pX + \frac{p+q}{4}$ assumes prime values at $k = 0, 1, \ldots, \frac{p+q}{4} - 2$.

Stark and Baker determined the imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ (with $d<0$ and d square-free) that have class number 2. According to their types, they are:

(i) $d=-6,-10,-22,-58$

(ii) $d=-5,-13,-37$

(iii) $d=-15,-35,-51,-91,-115,-123,-187,-235,-267,-403,-427.$

With these values of d one obtains optimal prime-producing polynomials.

In particular, $2X^2+29$ is an optimal prime-producing polynomial, with prime values at $k=0,1,\ldots,28$; it corresponds to the field $\mathbb{Q}(\sqrt{-58})$, which has class number 2.

- 29 is the number of distinct topologies on a set with 3 elements. Let τ_n denote the number of topologies on a set with n elements; thus $\tau_1=1$ and $\tau_2=2$. One knows the values of τ_n for $n\leq 9$ (Radoux, 1975).

Approaching the thirties, the age of confidence, life was smiling. 29 was the first twin prime age I reached since I became a mathematician by profession, so I select the number

30

- At this age I was in Bahia Blanca, Argentina, preparing a book which has, I believe, the distinction of being the southern-most published mathematical book. (At least this is true for books on ordered groups—but mine is not the northern-most published book on the subject.)
- There is only one primitive pythagorean triangle with area equal to its perimeter; namely $(5,12,13)$, with perimeter 30.
- 30 is the largest integer d such that if $1<a<d$ and $\gcd(a,d)=1$, then a is a prime. Other numbers with this property are 3, 4, 6, 8, 12, 18, and 24. This was first proved by Schatunowsky in 1893 and, independently, by Wolfskehl in 1901. (Wolfskehl is the rich mathematician who donated 100,000 golden marks to be given to the author of the first proof of Fermat's last theorem to be published in a recognized mathematical journal.)

This result has an interpretation as follows. Given $d>1$ and a, $1\leq a<d$, $\gcd(a,d)=1$, by Dirichlet's theorem, there exist infinitely many primes of the form $a+kd$ $(k\geq 0)$. Let $p(a,d)$ be the smallest such prime, and let

$$p(d)=\max\{p(a,d)\,|\,1\leq a<d, \gcd(a,d)=1\}.$$

If $d>30$, then $p(d)>d+1$. In particular,

$$\liminf\frac{p(d)}{d+1}>1.$$

Pomerance has shown:

$$\liminf\frac{p(d)}{\varphi(d)\log d}\geq e^{\gamma}$$

where $\varphi(d)$ is Euler's totient of d and γ is the Euler–Mascheroni constant.

On the other hand, as shown by Linnik, for d sufficiently large, $p(d)\leq d^L$, where L is a constant. Heath Brown showed that $L\leq 5.5$.

32

- 32 is the smallest integer n such that the number γ_n of groups of order n (up to isomorphism) is greater than $n : \gamma_{32} = 51$.

  I hate the number 32. At 32 degrees Fahrenheit, water becomes ice and snow begins to fall. Let us change the subject!

Older people remember best the events of their youth and those of the more recent past. I haven't forgotten anything I did not want to forget, so I could let you know about all the years 33, 34, ... But I would rather concentrate on the 60's.

60

- 60 was the base of numeration in the counting system of the Sumerians (ca. 3500 BC). Today we still use the sexagesimal system in astronomy and in the subdivisions of the hour.
- 60 is a *highly composite number*. Such numbers were introduced and studied by Ramanujan (1915): The natural number n is *highly composite* if $d(n) > d(m)$ for every m, $1 \le m < n$, where $d(n) =$ number of divisors of n. Thus $d(60) = d(2^2 \cdot 3 \cdot 5) = 3 \cdot 2 \cdot 2 = 12$. The smallest highly composite numbers are

$$2, 4, 6, 12, 24, 32, 48, 60, 120, 180, 240, 360, 720, 840, \ldots .$$

- 60 is a *unitarily perfect number*, which I now define. A number d is a *unitary divisor of* n if $d | n$ and $\gcd(d, n/d) = 1$; n is *unitarily perfect* if

$$n = \sum \{ d \mid 1 \le d < n,\ d \text{ unitary divisor of } n \}.$$

  Unitary divisors of 60 are 1, 3, 4, 5, 12, 15, 20 and their sum is indeed 60.

  Conjecture: There exist only finitely many unitarily perfect numbers.

  The only known unitarily perfect numbers are

$$6, 60, 90, 87360, \quad \text{and} \quad 2^{18} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313.$$

- 60 is the number of straight lines that are intersections of the pairs of planes of the faces of a dodecahedron.
- 60 is the order of the group of isometries of the icosahedron. This is the alternating group on 5 letters. It is the non-abelian simple group with the smallest order. The simple groups have been classified—a great achievement! There are 18 infinite families:
  - cyclic groups of prime order;
  - alternating groups A_n, with $n \ge 5$;
  - six families associated to the classical groups;
  - ten families associated to Lie algebras (discovered by Dickson, Chevalley, Suzuki, Ree, and Steinberg).

  There are also 26 "sporadic" groups, which don't belong to the above families. The sporadic groups with the largest order if Fischer's monster, which has

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \ge 8 \cdot 10^{53}$$

  elements.

61

- A curiosity: Let $k \geq 0$, and let $a_1, \ldots, a_k$, x, y be digits. If the number (in decimal notation)

$$a_1 a_2 \ldots a_k\, xyxyxyxyxy$$

is a square, then $xy = 21$, 61, or 84. Examples:

$$17392885161616161 = 131882088 1^2; \quad 258932382121212121 = 508853989^2.$$

- The Mersenne number $M_{61} = 2^{61} - 1$ is a prime. Today there are 37 known prime Mersenne numbers $M_p = 2^p - 1$, namely, those with

$$\begin{aligned} p = {} & 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, \\ & 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, \\ & 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, \\ & \text{and } 3021227. \end{aligned}$$

$2^{3021227} - 1$ is also the largest prime known today.

62

This number is remarkable for being so uninteresting. As a matter of fact, suppose that, for some reason or another, there is some number that is not remarkable. Then there is the smallest non-remarkable number, which is therefore remarkable for being the smallest non-remarkable number.

But this is just another example of Russell's paradox...

63

- This number appears in a cycle associated with *Kaprekar's algorithm* for numbers with 2 digits. This algorithm, for numbers with k digits, goes as follows: Given k digits $a_1 \ldots a_k$, not all equal, with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$, consider two numbers formed using these digits: $a_1 a_2 \ldots a_k$ and $a_k a_{k-1} \ldots a_1$. Compute their difference, and repeat the process with the k digits so obtained.

  Kaprekar's algorithm for 2, 3, 4, and 5 digits leads to the following fixed points or cycles.

  2 digits → cycle 63-27-45-09-81

  3 digits → 495

  4 digits → 6174

  5 digits → one of the three cycles: 99954-95553
98532-97443-96642-97731
98622-97533-96543-97641

  Example: $\{3, 5\}$: $53 - 35 = 18$, $81 - 18 = 63$, $63 - 36 = 27$, $72 - 27 = 45$, $54 - 45 = 09$, $90 - 09 = 81$.

- 63 is the unique integer $n > 1$ such that $2^n - 1$ does not have a *primitive prime factor*. Explanation: If $1 \leq b < a$, with $\gcd(a, b) = 1$, consider the sequence of binomials $a^n - b^n$ for $n \geq 1$. The prime p is a *primitive prime factor* of $a^n - b^n$ if $p | a^n - b^n$ but $p \nmid a^m - b^m$ if $1 \leq m < n$.

Zsigmondy proved, under the above assumptions, that every binomial $a^n - b^n$ has a primitive prime factor, except in the following cases:

(i) $n = 1$, $a - b = 1$;
(ii) $n = 2$, a and b odd, and $(a + b)$ a power of 2;
(iii) $n = 6$, $a = 2$, $b = 1$.

This theorem has many applications in the study of exponential diophantine equations; see [4]. Explicitly, when $a = 2$ and $b = 1$, the sequence is:

$$1, 3, 7, 15 = 3\cdot 5, 31, 63 = 3^2\cdot 7, 127, 257, 511, 1023 = 3\cdot 11\cdot 31, \ldots$$

64

64 is almost 65, a number I hated to reach, but which nevertheless has many interesting features.

65

- 65 is the smallest number that is the sum of 2 squares of natural numbers in 2 different ways (except for the order of summands):

$$65 = 8^2 + 1^2 = 7^2 + 4^2.$$

Recall Fermat's result: n is a sum of 2 squares if and only if for every prime $p \equiv 3 \pmod 4$, $v_p(n)$ is even. (Here $v_p(n)$ denotes the p-adic value of n, that is $p^{v_p(n)} \mid n$ but $p^{v_p(n)+1}$ does not divide n.) The following formula gives the number

$$r(n) = \#\{(a, b) \mid 0 \le b \le a \text{ and } n = a^2 + b^2\}.$$

For each $d \ge 1$, let

$$\chi(d) = \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } d \text{ is odd,} \\ 0 & \text{if } d \text{ is even.} \end{cases}$$

Let $R(n) = \sum_{d|n} \chi(d)$. Then

$$r(n) = \begin{cases} \dfrac{R(n)}{2} & \text{if } R(n) \text{ is even,} \\ \dfrac{1 + R(n)}{2} & \text{if } R(n) \text{ is odd.} \end{cases}$$

Example: $65 = 5\cdot 13$ has divisors 1, 5, 13, 65 and $R(65) = \sum_{d|65} \chi(d) = 4$, so $r(65) = 2$.

- 65 is the smallest hypotenuse common to two pythagorean triangles. This follows from the parametrization of the sides of pythagorean triangles: If $0 < x, y, z$, with y even and $x^2 + y^2 = z^2$, then there exist a and b, $1 \le b < a$, such that

$$x = a^2 - b^2; \quad y = 2ab; \quad z = a^2 + b^2.$$

Moreover the triangle is primitive (i.e., $\gcd(x, y, z) = 1$) if and only if $\gcd(a, b) = 1$. From $65 = 8^2 + 1^2 = 7^2 + 4^2$ one gets the pythagorean triangles (63, 16, 65) and (33, 56, 65).

- A curiosity: 65 is the only number with 2 digits d, e, $0 \leq e < d \leq 9$, such that $(de)^2 - (ed)^2 = \square$, a square. Indeed, $65^2 - 56^2 = 33^2$, and the uniqueness follows from the parametrization indicated above.
- 65 is also a remarkable number of the *second kind*, that is, it counts the number of remarkable numbers satisfying some given property. In the present case, 65 is perhaps the number of Euler's *numeri idonei*. I say "perhaps" because there is still an open problem, and instead of 65 there may eventually exist 66 such numbers.

Numeri idonei

What are these *numeri idonei* of Euler? Also called *convenient numbers*, they were used conveniently by Euler to produce prime numbers.

Now I'll explain what the *numeri idonei* are. Let $n \geq 1$. If q is an odd prime and there exist integers x, $y \geq 0$ such that $q = x^2 + ny^2$, then:

(i) $\gcd(x, ny) = 1$;
(ii) if $q = x_1^2 + ny_1^2$ with integers x_1, $y_1 \geq 0$, then $x = x_1$ and $y = y_1$.

We may ask the following question. *Assume that q is an odd integer, and that $q = x^2 + ny^2$, with integers $x, y \geq 0$, such that conditions (i) and (ii) above are satisfied. Is q a prime number?*

The answer depends on n. If $n = 1$, the answer is "yes," as Fermat knew. For $n = 11$, the answer is "no": $15 = 2^2 + 11 \cdot 1^2$ and conditions (i) and (ii) hold, but 15 is composite. Euler called n a *numerus idoneus* if the answer to the above question is "yes."

Euler gave a criterion to verify in a finite number of steps whether a given number is convenient, but his proof was flawed. Later, in 1874, Grube found the following criterion, using in his proof results of Gauss, which I will mention soon. Thus, n is a convenient number if and only if for every $x \geq 0$ such that $q = n + x^2 \leq \frac{4n}{3}$, if $q = rs$ and $2x \leq r \leq s$, then $r = s$ or $r = 2x$.

For example, 60 is a convenient number, because

$$\begin{aligned} 60 + 1^2 &= 61\bigstar, \\ 60 + 2^2 &= 64 = 4 \cdot 16 = 8 \cdot 8, \\ 60 + 3^2 &= 69\bigstar, \\ 60 + 4^2 &= 76\bigstar \end{aligned}$$

and the numbers marked with a $\bigstar$ do not have a factorization of the form indicated.

Euler showed, for example, that 1848 is a convenient number, and that

$$q = 18518809 = 197^2 + 1848 \cdot 100^2$$

is a prime number. At Euler's time, this was quite a feat.

Gauss understood convenient numbers in terms of his theory of binary quadratic forms. The number n is convenient if and only if each genus of the form $x^2 + ny^2$ has only one class.

Here is a list of the 65 convenient numbers found by Euler:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 21, 22, 24, 25, 28, 30, 33,
37, 40, 42, 45, 48, 57, 58, 60, 70, 72, 78, 85, 88, 93, 102, 105, 112, 120,
130, 133, 165, 168, 177, 190, 210, 232, 240, 253, 273, 280, 312, 330, 345,
357, 385, 408, 462, 520, 760, 840, 1320, 1365, 1848.

Are there other convenient numbers? Chowla showed that there are only finitely many convenient numbers; later, finer analytical work (for example, by Briggs, Grosswald, and Weinberger) implied that there are at most 66 convenient numbers.

The problem is difficult. The exclusion of an additional *numerus idoneus* is of a kind similar to the exclusion of a hypothetical tenth imaginary quadratic field (by Heegner, Stark, and Baker), which I have already mentioned.

An extraordinary conjunction

If your curiosity has not yet subsided, I was struck in 1989, in Athens, at the occasion of my "Greek Lectures on Fermat's Last Theorem," by an extraordinary conjunction of numbers. Once in a lifetime, and not to be repeated before...

At that year, my wife's age and my age were 59 and 61—twin primes (but we are not twins); at that same year, we had been married 37 years—the smallest irregular prime. If you are still interested, Kummer had proved that Fermat's last theorem is true for all odd prime exponents p that are regular primes. These are the primes p that do not divide the class number of the cyclotomic field generated by the p^{th} root of 1. Kummer also discovered that 37 is the smallest irregular prime. Pity that 1989 (the year of my Athens lecture) is not a prime.

So you are challenged to find the next occurrence of numbers like 37, 59, 61, but in a prime-numbered year.

Notes This paper on remarkable numbers would not have been possible were it not for the very original book by F. Le Lionnais, *Les Nombres Remarquables*, published in 1983 by Hermann, in Paris.

François Le Lionnais was not a mathematician by profession, but rather a scientific writer, and as such, very well informed. His book *Les Grands Courants de la Pensée Mathématique* is very engrossing to read even today. Just after the war he gathered in this book the ideas of several young French mathematicians—still little known at that time—who would soon rise to the pinnacle. An English translation and the original are available in good libraries. I have an autographed copy of the book on remarkable numbers, where Le Lionnais thanked me for calling his attention to the number 1093. You may read about this number in my article 1093, *Math. Intelligencer* **5** (1983), 28–34.

Another book of the same kind, which served me well, is: D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers*, Penguin, London, UK, 1986.

For results on algebraic numbers, nothing is easier for me than to quote my own book **[3]**, to appear in a new edition at Springer-Verlag. For *numeri idonei*, see **[1]**. Concerning primitive factors of binomials, see **[4]**. On prime numbers, Fibonacci numbers and similar topics, see **[5]**. For further reference, see **[2]**.

The following list of references is, it goes without saying, incomplete.

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The Newest Inductee in the Number Hall of Fame

COLIN C. ADAMS
Williams College
Williamstown, MA 01267

"Rational and irrational, algebraic and transcendental, whole, natural, positive, negative, even, odd, prime, cardinal, ordinal, yes even p-adic numbers: It gives me great pleasure to welcome you all to these induction ceremonies at the Number Hall of Fame here in Canton, Ohio. It's truly gratifying to see such a tremendous and diverse turnout for this event. How wonderful it is that we can forget our differences, put aside our animosities, and come together to honor those amongst us who have truly achieved a purpose in life above and beyond our appearance on ledgers, computer screens, and cash registers around the globe.

The vast throng of numbers here today must reflect, at least in part, the rarity of these occasions. We are not that type of Hall of Fame where any number eventually makes the cut. Not like the Hall of Fame of Physical Constants, where a number is embarrassed not to eventually appear in their list. No, as you can see by the small set of numbers sitting up on the dais, including most of the past recipients of the award, this honor is bequeathed on only a select few.

Let me quickly introduce you to our past recipients. On the far left we have the first inductee into the Number Hall of Fame, someone I need say little about, as everyone knows her well. Yes, it's the delightful number One. How could anyone be less than charmed by her ability to multiply a number without altering it?

Sitting to the left of One is the incomparable e, that paradoxical sorcerer who when put to the power x becomes his own derivative. What a pleasure it is to have you here today, sir.

Just to the left of e, we have that mandarin enigma, the Buddha of math, Mr. Zero. Please, don't get up. It's not a good idea after your triple bypass.

Next to Mr. Zero, we have the ebullient Two. As he is all too fond of saying, "You need Two to tango." Seated next to Two is his son, $\sqrt{2}$, who with his simple and forthright manner, hardly deserves to be called irrational.

Seated in the empty chair to the left of $\sqrt{2}$ is his imaginary friend i. Please, no snickering from you Bernoulli numbers in the back. Having an imaginary friend doesn't necessarily mean you're a couple million digits past rational, even if in this particular case, it happens to be true.

And at the end sits $\frac{1+\sqrt{5}}{2}$, that icon of Greek aesthetics, known affectionately as the Golden Ratio. Her attendance at any event brings an elegance and refinement to the proceedings.

Several of our inductees could not be with us today, including Euler's constant, better known as $\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln(N)\right)$ or .5772157... to her friends. Unfortunately, she twisted her ankle at last night's pre-ceremony ball while doing the Watusee.

Finally, standing right next to me, we have someone who needs no introduction of her own, but who has agreed to give the introduction for our new inductee. Ladies and Gentlemen, I am very pleased to present π."

"Well thank you, Six, for such a warm welcome from the perfect host. I am greatly honored to have the privilege of introducing our new inductee. Perhaps I should begin by explaining why I was chosen to make these remarks. Many of you know me as the area of a circle of radius one, or half the perimeter of that circle. You may be familiar with my appearance with my dear friend e in normal distributions as defined in probability. I have been lucky in my career to have more than one role to play. But perhaps you are unfamiliar with another capacity where I have been able to contribute in my own small way, really just over the last hundred or so years, since the discovery of hyperbolic geometry.

Let me give a little background on that, as I realize some of you numbers haven't been paying a lot of attention to recent advances in geometry. Discovered in the mid 1800's, hyperbolic geometry's existence was the proof that Euclid's parallel postulate was independent of the other axioms of geometry, as here was a geometry that did not satisfy it. Its properties were so extraordinary that Gauss knew about it for 30 years, but kept mum, for fear of denigrating verbal abuse from his colleagues.

Let me begin my description of hyperbolic geometry with the upper-half-plane model of the hyperbolic plane, which is relatively easy to visualize. I brought along some graphics to help us along. May we have the first slide, please? In FIGURE 1, we

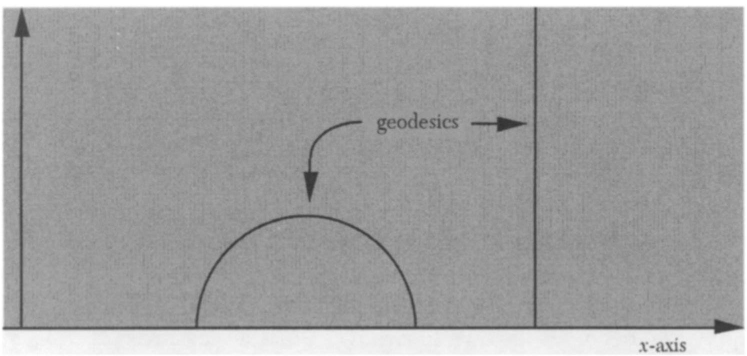

FIGURE 1
The upper-half-plane model of the hyperbolic plane.

see the points in the hyperbolic plane, namely all the points in the x-y plane having positive y-coordinate. The curves that play the role of straight lines in this geometry, what we call *geodesics*, will be the vertical half-lines, and the semi-circles perpendicular to the x-axis. By calling such a curve a geodesic, we simply mean that the shortest path between two points on such a curve is the sub-arc of that curve with those two points as endpoints.

We measure the length of a curve by integrating $1/y$ along it. So the distance between two points is $\int_\gamma \frac{1}{y}\,ds$, where γ is the geodesic path from the one point to the other.

Now, as you know, in the Euclidean plane, the sum of the angles of a triangle, measured in radians, is always exactly equal to me, that is to say π. Moreover, a triangle can have any positive area. In particular, there is no upper limit to the size of a triangle.

But in the hyperbolic plane, we encounter a very different world indeed. The area of a triangle is given by $\pi-(\alpha+\beta+\gamma)$, where α, β, and γ are the angles of the triangle given in radians. I will demostrate this in just a bit, but first let me point out the implications. The angles of the triangle determine the area. This means there can be no scaling up or down as there is in Euclidean space. A triangle with specified angles only comes in one size. Moreover, since the area of a triangle must be positive, the sum of the angles of a hyperbolic triangle must be strictly less than π. Clearly, hyperbolic space is not just a slight variation on Euclidean space. The most basic tenets of geometric behavior, which we consider interwoven into the fabric of Euclidean geometry, become nonsensical in hyperbolic space.

To see that the area of a hyperbolic triangle is given by $\pi-(\alpha+\beta+\gamma)$, let us first compute the area of a triangle having one vertex with angle zero. Now, in order to have an angle of zero, that vertex of the triangle must be pulled all the way out to a point on the boundary of hyperbolic space, either on the x-axis or out the positive y-axis at $\{\infty\}$. Although this means the vertex will be missing from the triangle, since the boundary of hyperbolic space is not a part of hyperbolic space, we can still determine the area of this slightly cropped triangle. It is convenient to choose the vertex with angle zero to be the one centered at $\{\infty\}$, and the bottom edge of the triangle to appear on the unit circle, so our triangle appears as in FIGURE 2. All triangles with a single angle of zero are equivalent to a triangle like this one.

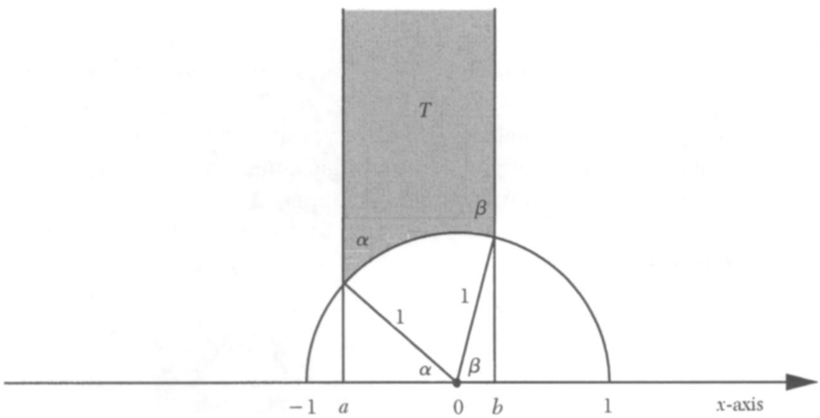

FIGURE 2
A hyperbolic triangle with one angle 0.

The hyperbolic area A of this triangle T is obtained by integrating $1/y^2$ over the triangle. Since $a=\cos(\pi-\alpha)$ and $b=\cos\beta$, we obtain:

$$A=\int_a^b\int_{\sqrt{1-x^2}}^{\infty}\frac{1}{y^2}\,dy\,dx=\arcsin(b)-\arcsin(a)$$
$$=\pi/2-\beta-(\alpha-\pi/2)=\pi-(\alpha+\beta).$$

A triangle T_1 with three non-zero angles can always be thought of as a sub-triangle of a triangle such as T, by extending one edge of T_1 off to infinity and pulling a vertex of T_1 on that edge off to infinity, as in FIGURE 3. The area of triangle T_1 will be the difference of the areas of the two triangles T and T_2, each of which has a vertex with angle zero.

Therefore $A(T_1)=A(T)-A(T_2)=\pi-\alpha-\beta-(\pi-(\beta_2+\gamma_2))=\pi-(\alpha+\beta_1+\gamma_1)$. This completes the proof.

You will notice right away that the smaller the angles, the bigger the area. So if we want a triangle with as large an area as possible, we should take the angles to be as small as possible. Actually, we would like to take a triangle with all angles equal to

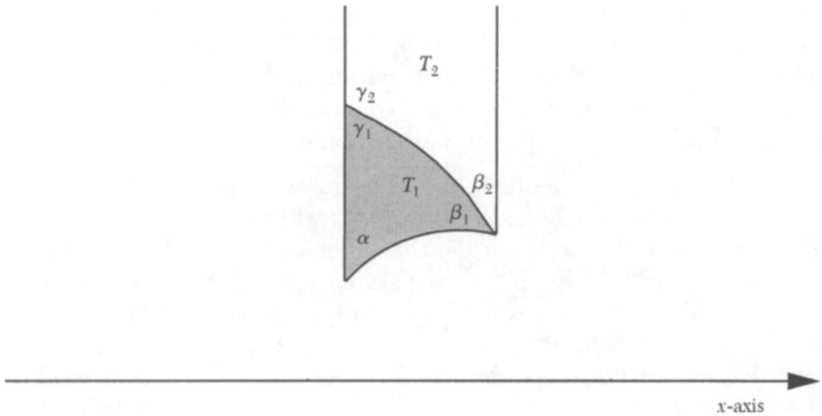

FIGURE 3
$A(T_1) = A(T) - A(T_2)$.

zero. If we allow ourselves to put all the vertices of the triangle on the boundary of hyperbolic space, we construct a triangle with all angles equal to zero, and area equal to π. Such a triangle, missing its three vertices, is called an *ideal triangle*. Note that we needn't take any of the vertices up at $\{\infty\}$ if we don't want to. (See FIGURE 4.)

So to make the point you have all been patiently waiting for, besides being the area of a circle of radius 1 and half the perimeter of that same circle, I am also the area of any ideal triangle in hyperbolic 2-space, which happens to be the triangle of greatest area in hyperbolic space. But what does this have to do with our newest inductee into the Number Hall of Fame?

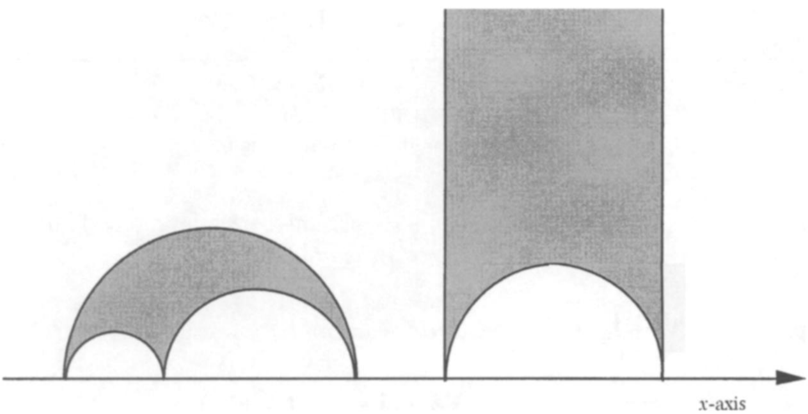

FIGURE 4
Ideal triangles in the upper-half-plane model.

My esteemed colleagues of all persuasions, I am proud to introduce to you the *Gieseking constant*. Little known outside of hyperbolic circles, G.C. is an up-and-comer who will be playing an important role for many years to come. Let me define her for you in an analogy to my own realization as the area of an ideal triangle in the hyperbolic plane. However, now we will step up a dimension to hyperbolic 3-space. Let us work in the upper-half-space model of hyperbolic 3-space, where geodesics correspond to vertical half-lines and semi-circles perpendicular to the boundary. Here,

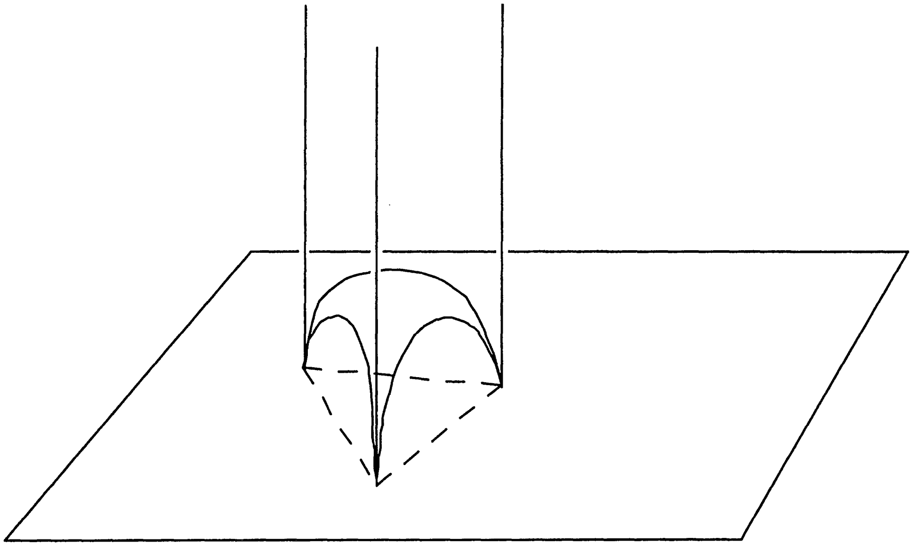

FIGURE 5
An ideal tetrahedron in the upper-half-plane model of hyperbolic 3-space.

we also have geodesic planes that correspond to hemispheres perpendicular to the boundary and vertical half-planes. And instead of an ideal triangle, we will look at what is called an ideal tetrahedron. To construct such an object, we want four faces, each individually an ideal triangle contained within a geodesic plane, such that any two share an edge which is itself a geodesic, and any three meet at a single ideal vertex. For convenience, we will choose one of the vertices to occur up at the top of the positive z-axis, at $\{\infty\}$, as in FIGURE 5.

Such an ideal tetrahedron has a variety of interesting properties. First I will demonstrate that the sum of the three dihedral angles around a vertex is equal to me, which is to say π. How to see this? Slice the top off the tetrahedron with a horizontal plane, called a *horosphere*. Then the dihedral angles of the three vertical edges form the three angles of a Euclidean triangle in that plane, which must therefore sum to π. Since the vertex at $\{\infty\}$ is no different from any other vertex, but just appears so in this model, the same fact will hold for all of the vertices.

In addition, any two opposite dihedral angles of the tetrahedron are equal to one another. Here is a quick trick to convince yourselves that this is true. Take two opposite edges. Then there is a unique geodesic perpendicular to both, as in FIGURE 6. This will contain the shortest geodesic arc from one to the other. Such a shortest arc always exists between two geodesics in hyperbolic space, assuming they do not intersect in either an interior point or their endpoints. Now rotate the entirety of the ideal hyperbolic tetrahedron 180 degrees about the geodesic. Each of the two opposite edges will be sent back to itself, but with its endpoints interchanged. Thus, this rotation permutes the vertices of the tetrahedron and therefore sends the entire ideal tetrahedron back to itself. Moreover, it switches the other two pairs of opposite edges, and they must therefore have the same dihedral angles. Since we can do this for any of the three pairs of opposite edges, the dihedral angles on each opposite edge must be the same.

Thus three dihedral angles around a single vertex determine all the dihedral angles. This is enough to determine the tetrahedron itself. Since these three angles sum to π, there are actually only two degrees of freedom for ideal tetrahedra, say α and β.

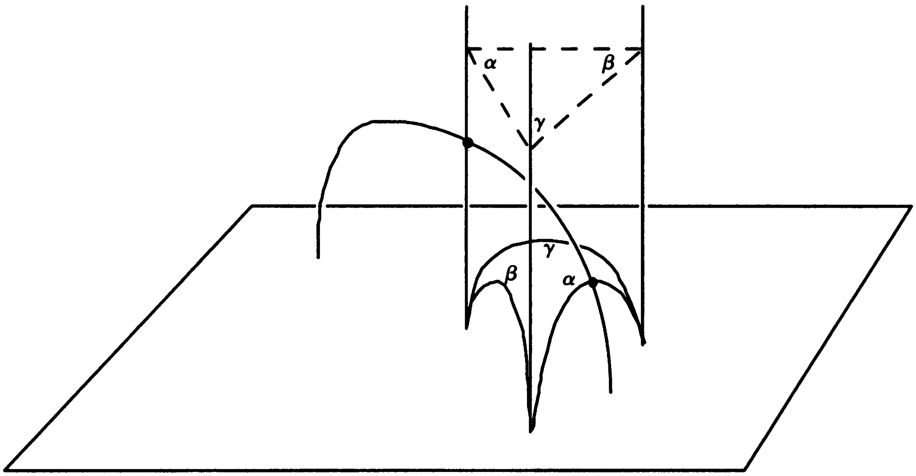

FIGURE 6
Opposite dihedral angles are equal.

By now you may be wondering, in analogy with our discussion of the area of hyperbolic triangles in the hyperbolic plane, what is the largest possible volume of a tetrahedron in hyperbolic space? Clearly, if we take a tetrahedron with vertices occurring inside hyperbolic space, we can always find one a little bigger, by pulling one of the vertices out a little farther toward infinity. But if we also allow tetrahedra with ideal vertices, then the maximum volume will occur for a tetrahedron with all ideal vertices, which is to say an ideal tetrahedron.

Remember that for ideal triangles, all such had the same area. But for ideal tetrahedra, the volume is not always the same. It depends on the dihedral angles α, β, and γ. For what ideal tetrahedron is the volume the largest possible? We will see that it is the regular one, with all dihedral angles $\pi/3$. That's not too surprising, since if a maximum exists, we would expect it to occur at a tetrahedron with a lot of symmetry.

First, let's find a formula that gives the volume of an ideal tetrahedron in terms of its determining angles. We need to integrate $1/z^3$ over the entirety of the tetrahedron. Let's choose our ideal tetrahedron to have one vertex up at $\{\infty\}$ and the bottom

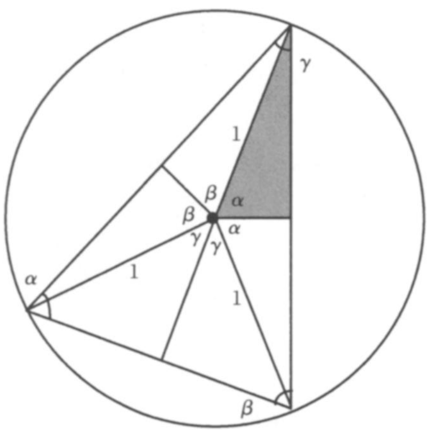

FIGURE 7
Projecting an ideal tetrahedron to the x-y plane.

hemispherical face on the hemisphere given by $z = \sqrt{1 - x^2 - y^2}$. Looking down from way high up on the positive z-axis, the tetrahedron looks like a triangle inscribed in the unit circle. We can orient the triangle so that one edge is perpendicular to the x-axis. A triangle inscribed in the unit circle always has its center at the center of the circle, so we can cut this triangle up into six right triangles as in FIGURE 7. Note that the labeling of the angles at the center is a famous fact from geometry that follows when one considers the three isosceles triangles that together make up the inscribed triangles.

First, we will find the volume of that part of the tetrahedron above a single one of these triangles, the one that is shaded in the figure. Let's call that volume $V(\alpha)$.

We need to form a triple integral as z ranges from $\sqrt{1 - x^2 - y^2}$ to ∞, y ranges from 0 to $x \tan \alpha$, and x ranges from 0 to $\cos \alpha$. Thus

$$\begin{aligned} V(\alpha) &= \int_0^{\cos a} \int_0^{x \tan a} \int_{\sqrt{1-x^2-y^2}}^{\infty} \frac{1}{z^3}\, dz\, dy\, dx \\ &= \int_0^{\cos \alpha} \int_0^{x \tan \alpha} \frac{1}{2(1 - x^2 - y^2)}\, dy\, dx \\ &= \int_0^{\cos \alpha} \int_0^{x \tan \alpha} \frac{1}{4\sqrt{1-x^2}} \left(\frac{1}{\sqrt{1-x^2} - y} + \frac{1}{\sqrt{1-x^2} + y} \right) dy\, dx \\ &= \int_0^{\cos \alpha} \frac{1}{4\sqrt{1-x^2}} \ln\left(\frac{\sqrt{1-x^2} + x \tan \alpha}{\sqrt{1-x^2} - x \tan \alpha} \right) dx \\ &= \int_0^{\cos \alpha} \frac{1}{4\sqrt{1-x^2}} \ln\left(\frac{\sqrt{1-x^2} \cos \alpha + x \sin \alpha}{\sqrt{1-x^2} \cos \alpha - x \sin \alpha} \right) dx. \end{aligned}$$

Substituting in $x = \cos \theta$, so $\sqrt{1 - x^2} = \sin \theta$ and $dx = -\sin \theta\, d\theta$, we obtain

$$\frac{1}{4} \int_{\alpha}^{\pi/2} \ln\left(\frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)} \right) d\theta.$$

Letting $u = \theta - \alpha$, the integral becomes

$$V(\alpha) = \frac{1}{4} \int_0^{\pi/2 - \alpha} \ln\left(\frac{\sin(u + 2\alpha)}{\sin(u)} \right) du.$$

We will use this formula as is. Thus, the volume of the entire ideal tetrahedron is

$$\mathrm{Vol(T)} = 2\,V(\alpha) + 2\,V(\beta) + 2\,V(\gamma).$$

Now, of course, this is not the most satisfying of formulas, as one would prefer a closed form made up of elementary functions rather than a formula involving integrals of natural logs of trig functions. However, for our purposes, it will suffice. Our goal is now to determine the angles α, β, and γ that will give the maximum volume for this ideal tetrahedron. To that end we will use Lagrange multipliers, with Vol(T) a function depending on α, β, and γ, and constraint $\alpha + \beta + \gamma - \pi = 0$. We find that at the maximum,

$$V'(\alpha) = V'(\beta) = V'(\gamma) = \lambda.$$

To compute $V'(\alpha)$, we need to take the derivative of an integral with respect to a variable that appears in a limit of integration as well as in the integrand. Using the chain rule and the fundamental theorem of calculus, we find

$$V'(\alpha) = -\frac{1}{4}\ln\left(\frac{\sin(\pi/2+\alpha)}{\sin(\pi/2-\alpha)}\right) + \frac{1}{4}\int_0^{\pi/2-\alpha} \frac{\cos(u+2\alpha)}{\sin(u+2\alpha)} 2\,du$$

$$= -\frac{1}{4}\ln(1) + \frac{1}{2}\int_{2\alpha}^{\pi/2+\alpha} \frac{\cos(w)}{\sin(w)}\,dw$$

$$= 0 + \frac{1}{2}\ln\left(\frac{\sin(\pi/2+\alpha)}{\sin(2\alpha)}\right)$$

$$= \frac{1}{2}\ln\left(\frac{\cos\alpha}{2\sin\alpha\cos\alpha}\right) = -\frac{1}{2}\ln(2\sin\alpha).$$

Thus, in order that $V'(\alpha) = V'(\beta) = V'(\gamma)$, it must be that $\sin\alpha = \sin\beta = \sin\gamma$. If it is not true that $\alpha = \beta = \gamma$, then it must be that one of the angles is π, and the other two are 0, but not surprisingly, this yields a minimum volume of 0. Thus, the only possibility is $\alpha = \beta = \gamma = \pi/3$, and the regular ideal tetrahedron with all dihedral angles equal to $\pi/3$ is the hyperbolic tetrahedron of maximal volume.

As I gaze out upon you, I can see that your curiosity is piqued. Exactly what *is* that volume? How big can a tetrahedron be in hyperbolic space? The volume of the ideal regular tetrahedron is given by $6V(\pi/3)$. Unfortunately, we cannot directly integrate to find $V(\pi/3)$. But by numerical integration, $V(\pi/3) = .16915\ldots$ and therefore the largest volume for a tetrahedron in hyperbolic space is $1.01494\ldots$.

That number, $1.01494\ldots$, is our esteemed guest, the newest member in the Number Hall of Fame. I am pleased to present to you V_0, also known as the Gieseking constant. She will make a few remarks."

"Oh, what an honor it is to be up here with such an august group. I am thrilled to be here. And it is appropriate that so many different kinds of numbers are present, because the honest truth is that I do not myself know to what group I belong. Am I rational, am I irrational? Am I algebraic or transcendental? I do not know the answers to these most basic questions about my true identity. In time, perhaps, I will know where I belong, but for now, I am a representative of all the diversity inherent in numbers, exempt from the systematic classification so worshipped in our times.

As π mentioned, I am the largest volume of a tetrahedron in hyperbolic 3-space, ideal or otherwise, and I occur exactly for the volume of an ideal regular tetrahedron. In fact, I am the eldest sibling in a family of numbers. My sisters and brothers are the volumes of the ideal regular n-simplices in hyperbolic n-space. They are also the maximal volumes for any n-simplex, ideal or otherwise, in hyperbolic n-space. This nontrivial fact was only proved in 1981 and appears in a paper by Haagerup and Munkholm (cf. **[2]**).

I should explain where I come by the name "Gieseking's constant." H. Gieseking was a mathematician around the turn of the last century. He realized that if one takes an ideal regular tetrahedron and glues up pairs of faces by appropriate hyperbolic isometries, one can create a hyperbolic manifold, appropriately called Gieseking's manifold. This manifold is interesting for a variety of reasons. First of all, it is not a compact manifold. Because it is constructed from an ideal tetrahedron, the missing vertices make it noncompact.

Second, it has a volume exactly equal to me, which is to say 1.01494.... So here is a noncompact manifold that still has finite total volume. Even though its arm reaches all the way out to infinity, the cross section of the arm shrinks exponentially to 0 in area, and the manifold still has finite volume.

Third, it is non-orientable. If we were all inside this manifold and you, 81, were to walk through a face of the tetrahedron, you would suddenly appear to us to be 18. You would be reversed right to left. Bit of a scary thought, isn't it? Oh, no insult intended, 18.

Fourth, it is double covered by the figure-eight knot complement. I won't go into details here, but this is a truly remarkable fact. It is one example of the tremendously important ties between hyperbolic geometry and so-called "low dimension topology" that were established by William Thurston in the late seventies and early eighties.

Finally, it is known that among all noncompact hyperbolic 3-manifolds, this particular one has the least volume (cf. **[1]**). So I am the least volume among all volumes of noncompact hyperbolic 3-manifolds, orientable or otherwise.

At any rate, I just wanted to thank you from the bottom of my decimal point for the great honor you have bestowed upon me. Now, I will turn the podium back over to Six."

"Well, thank you G.C., for those eloquent remarks. You know, numbers, she is too modest to say it, but she is also equal to $\frac{9\sqrt{3}\,\zeta_{Q[i\sqrt{3}]}(2)}{2\pi^2}$ where $\zeta_{Q[i\sqrt{3}]}(2)$ is the value of the Dedekind zeta function for the field $Q[i\sqrt{3}\,]$ at 2 (cf. **[4]**). Of course, this makes her quite intriguing to number theorists. Moreover, when divided by our friend π, she mysteriously produces the logarithm of the Mahler measure of the polynomial $1+x+y$ (cf. **[3]**). Her unexpected appearances in a variety of disparate mathematical locations make her a very enchanting and mysterious figure.

But I see from the amount of fidgeting in the audience that our time is up. We will follow these ceremonies with the traditional banquet and karaoke contest. I want to thank the organizing committee, which again consisted of the odd numbers less than 8. Not too surprisingly, the menu for the banquet is a repeat of last year's: prime rib, 3-bean salad, 7-grain bread, and 5-layer cake for dessert. This year, we ask that all integers greater than 999 and all decimal expansions please use the specially widened food line to the left, so we can avoid the congestion, antagonism, and subsequent chaos of previous years. I have been instructed to particularly request that the Fibonacci numbers behave themselves. And again, thanks for coming. See you all next year."

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The Crossing Number of $C_m \times C_n$: A Reluctant Induction

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1. Introduction

"Even a fool," remarked Paul Erdős in one of his many lectures, "can ask questions that the wisest man cannot answer." This statement is true for many areas of mathematics—perhaps nowhere more than in graph theory. The four-color theorem, proved more than a century after it was proposed, illustrates Erdős's point. Another example is Turán's "Brick Factory" Problem. Although it was thought for some years to have been solved, flaws in the proof were discovered nearly twenty years later, and it remains open today. In this article, we explore another combinatorial problem, one that is simple to state and would seem to be provable by induction, but that has been found to be tantalizingly difficult. It is a crossing number problem that can be stated roughly as follows: For a rectangular grid on a torus, is there a planar drawing of this graph that has fewer crossings than the one shown in FIGURE 1? Attempts to solve this

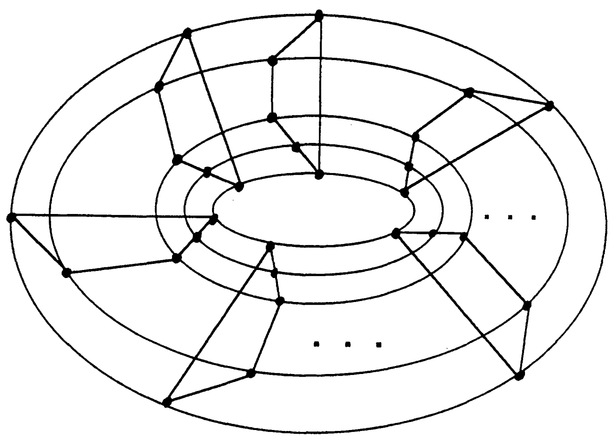

FIGURE 1
$C_m \times C_n$.

problem have used varying techniques, none of which has been entirely successful. Below we give a brief history of crossing numbers; then we present our problem, its history, and current status.

2. Origin of the crossing number problem

In 1977 the Hungarian mathematician Pal Turán wrote [19]:

> In July 1944 the danger of deportation was real in Budapest, and a reality outside Budapest. We worked near Budapest, in a brick factory. There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled

> trucks to the storage yards. All we had to do was to put the bricks on the trucks at the kilns, push the trucks to the storage yard, and unload them there. We had a reasonable piece rate for the trucks, and the work itself was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time which was rather precious to all of us (for reasons not to be discussed here). We were all sweating and cursing at such occasions, I too; but *nolens-volens* the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings? I realized after several days that the actual situation could have been improved, but the exact solution of the general problem with m kilns and n storage yards seemed to be very difficult and again I postponed my study of it to times when my fears for my family would end. (But the problem occurred to me again not earlier than 1952, at my first visit to Poland where I met Zarankiewicz. I mentioned to him my "brick factory" problem... .) This problem has... become a notoriously difficult unsolved problem.

Turán's statement puts the origin of the crossing number problem at mid-twentieth century. Graph theory itself is not much older. Although Euler's solution of the Königsburg bridge problem appeared in 1736, mathematics historians date the rise of modern graph theory to 1936. In that year appeared *Theorie der Endlichen und Unendlichen Graphen*, by the Hungarian mathematician Dénes König **[13]**. Blanche Descartes **[6]** light-heartedly emphasized this book's significance in a "Hymn for Graph Theorists":

> Graph Theory's one foundation
> Is König's famous book.
> It gives an explanation—
> If you will only look—
> Of cycles, nodes and edges
> And graphs complete, called K,
> And how to cross your bridges
> In an Eulerian way.

3. Basic notation and terminology

A *graph* G with n *vertices* (or *nodes*) and m *edges* consists of a *vertex set* $V(G)=\{v_1,\ldots,v_n\}$ and an *edge set* $E(G)=\{e_1,\ldots,e_m\}$, where each e_i is a two-element subset of $V(G)$. An edge $e=\{u,v\}$ is usually written uv. Vertices u and v are called *endpoints* of e. A *drawing* D of G in a plane P represents each vertex as a distinct point of P and each edge uv as an open arc containing no vertices, joining u to v in P. A *crossing* in a drawing is an intersection of two edges. In this paper, we require that a drawing must satisfy three further conditions: (i) no two edges that share an endpoint may cross, (ii) no two edges may cross more than once, and (iii) no three edges may cross at a single point. A graph is *planar* if it has a drawing with no crossings. Such a drawing is called a *plane drawing* of G. The *crossing number* $\nu(G)$ is the minimum number of crossings among all drawings of G in the plane. An *optimal* drawing has $\nu(G)$ crossings.

A graph's name may suggest its nature. For instance, a *path* P_n has distinct vertices $v_0, v_1, \ldots, v_n$ such that $v_i v_{i+1}$ is an edge, for $i = 0, \ldots, n-1$. Similarly, an *n-cycle* C_n has distinct vertices $v_1, \ldots, v_n$ such that all $v_i v_{i+1}$ are edges, as well as $v_n v_1$. The graph K_n with n vertices and an edge connecting each pair of vertices is called the *complete graph on n vertices*. A graph is *bipartite* if its vertex set comprises disjoint sets A and B, such that every edge has one endpoint in each set. If $|A| = m$ and $|B| = n$ the bipartite graph with mn edges is called *complete bipartite* and is denoted $K_{m,n}$.

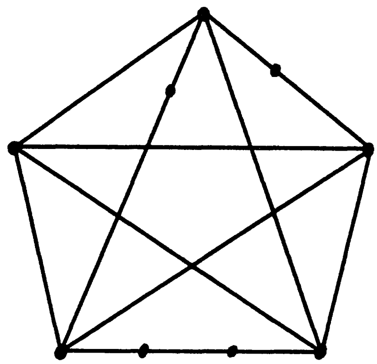

FIGURE 2
A subdivision of K_5.

In graph theory, as in other areas of mathematics, one derives new structures from old. For example, H is a *subdivision* of G if H can be obtained from G by successive operations of the following kind: Delete edge uv and add vertex w along with edges uw and wv. We regard G as a subdivision of itself, having performed zero operations of the required kind. FIGURE 2 shows a subdivision of K_5. For graphs G and C_n, the *Cartesian product graph* $G \times C_n$ is obtained by making n copies of G, then joining corresponding vertices in a cyclic fashion. A plane drawing of $P_1 \times C_4$ appears in FIGURE 3. This paper concerns $C_m \times C_n$. Since it is isomorphic to $C_n \times C_m$, we shall always write $C_m \times C_n$ with $m \leq n$.

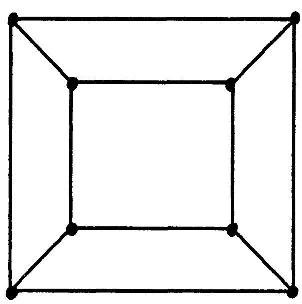

FIGURE 3
$P_1 \times C_4$.

4. Useful results and techniques

In a plane drawing of a graph G a *face* is any maximal region disjoint from G. For planar graphs the number of faces is determined by the number of vertices and edges as the following results indicate:

EULER'S FORMULA: *For a plane drawing of a connected graph with n vertices, m edges and f faces, $n - m + f = 2$.*

EDGE-VERTEX INEQUALITY FOR PLANAR GRAPHS: *Any planar graph G with n vertices ($n \geq 3$) and m edges satisfies $m \leq 3n - 6$. If, also, G has no triangles (3-cycles), then $m \leq 2n - 4$.*

For proofs, see **[21]**. Applying the edge-vertex inequality to K_5, which has five vertices and ten edges, we see that K_5 is not planar. Similarly, the triangle-free $K_{3,3}$ is not planar. These graphs embody the essence of non-planarity in the following sense: A theorem of Kuratowski **[14]** states that G is planar if and only if G contains no subdivision of either K_5 or $K_{3,3}$.

Many readers are familiar with the "Houses and Utilities" Puzzle, which asks whether it is possible to connect three houses each with three utilities without the utility lines crossing. If not, what is the least number of crossed utility lines? In the language of graph theory, the problem asks: What is $\nu(K_{3,3})$? Since $K_{3,3}$ is not planar, $\nu(K_{3,3}) \geq 1$. FIGURE 4 shows that $\nu(K_{3,3}) \leq 1$. Thus $\nu(K_{3,3}) = 1$. This illustrates a useful technique for proving $\nu(G) = n$: Argue that any drawing of G must have at least n crossings, then exhibit a drawing of G with n crossings.

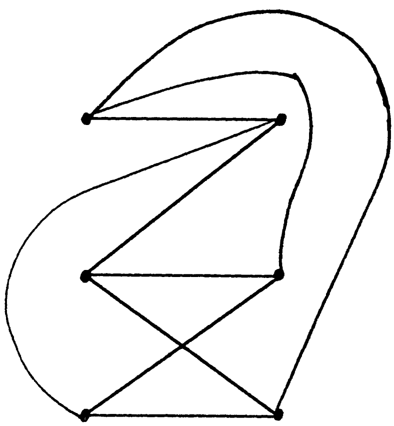

FIGURE 4
$K_{3,3}$ with 1 crossing.

5. Crossing number problems

In 1970, Erdős and Guy **[8]** observed, "Almost all questions that one can ask about crossing numbers remain unsolved." Today there are still only a few graphs for which crossing numbers have been established. Apart from elementary modifications of graphs with known crossing numbers, the only graphs whose crossing numbers are large and known are K_n for $3 \leq n \leq 10$, $K_{m,n}$ for $3 \leq m \leq 6$, $C_m \times C_n$ for $3 \leq m \leq 6$, and $C_7 \times C_7$. An interesting introduction to these and other crossing number problems is found in **[3]**.

FIGURE 3 shows that $\nu(P_1 \times C_4) = 0$. Indeed, it is easily seen that $\nu(P_m \times C_n) = 0$ for all m and n. However, establishing $\nu(C_m \times C_n)$ is quite a different problem. How does one determine $\nu(C_m \times C_n)$? Since graphs are finite structures, it seems that double induction should be effective. After all, $C_{m+1} \times C_n$ looks much like $C_m \times C_n$, and induction is often used in graph theory proofs. W. T. Tutte once noted **[20]**: "We... look at graphs, state... regularities as conjectural theorems, then try to prove those theorems for all graphs, even for those soaring out of sight. It works sometimes, usually by the grace of the principle of mathematical induction." But this is apparently not one of those times. For the values of m for which $\nu(C_m \times C_n)$ is known for all n, $\nu(C_m \times C_m)$ was established first. Then $\nu(C_m \times C_n)$ was established by induction on n. Induction on m has been so reluctant as to be nonexistent.

6. Early results on $\nu(C_m \times C_n)$

In this section we discuss the case $m = 3$. Here and in the sections to follow, we offer proof summaries for some results, to suggest the flavor of the methods used.

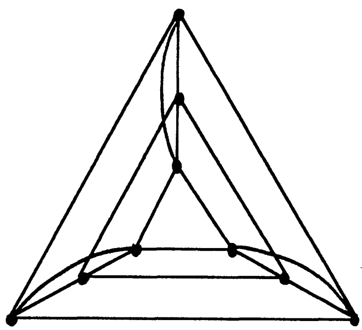

FIGURE 5
$C_3 \times C_3$ with 3 crossings.

In a 1973 paper, Harary, Kainen, and Schwenk **[11]** used a drawing such as FIGURE 5 in their proof that $\nu(C_3 \times C_3) = 3$. They then conjectured that an analogous drawing of $C_m \times C_n$ (FIGURE 1) would be optimal. More precisely:

The (m, n)*-conjecture*: *For* $m \leq n$, $\nu(C_m \times C_n) = (m-2)n$.

Not until 1978 was the $(3, n)$-conjecture verified. In their proof, Ringeisen and Beineke **[16]** used a pattern of three techniques which would be used repeatedly in subsequent results on crossing numbers for $C_m \times C_n$. The first technique is edge-coloring. The second identifies a situation that causes $\nu(C_m \times C_n)$ to be at least $(m-2)n$. The third is induction on n. In a crucial lemma, Ringeisen and Beineke used the first two techniques. From this lemma, the $(3, n)$-conjecture follows by an easy induction. (A proof summary for the lemma appears after Theorem $(3, n)$.)

THEOREM $(3,3)$. $\nu(C_3 \times C_3) = 3$.

Proof summary. FIGURE 5 shows that $\nu(C_3 \times C_3) \leq 3$. Since all edges of $C_3 \times C_3$ are essentially alike, if $\nu(C_3 \times C_3)$ were 1, removing an edge of $C_3 \times C_3$ would result in a planar graph. But FIGURE 6 shows that $C_3 \times C_3 - \{e\}$ contains a subdivision of K_5 and hence cannot be planar.

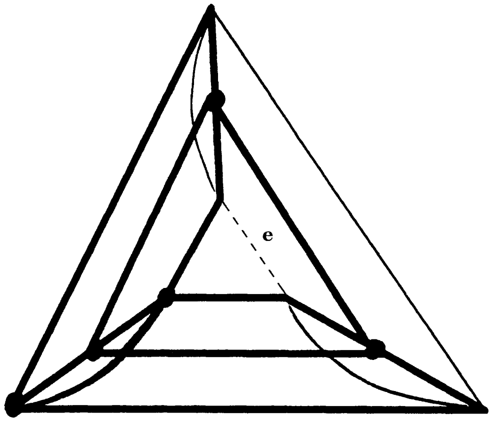

FIGURE 6
A subdivision of K_5 in $C_3 \times C_3 - \{e\}$.

Suppose next that $\nu(C_3 \times C_3) = 2$, and that D is an optimal drawing. In D, color red the concentric 3-cycles and blue the crossing 3-cycles. Is it possible for two red edges to cross? If so, the definition of a drawing implies that the crossing red edges must come from distinct triangles. In that case the two red triangles cross twice, allowing no further crossings in D. Hence the third red triangle must lie entirely inside or outside each of the other red triangles. Since blue triangles connect corresponding vertices on all three red triangles, some blue triangle must pass from the inside to the outside of the original red triangle, producing a third crossing. Thus it is impossible for two red edges to cross. A crossing of two blue edges is similarly impossible.

If $\nu(C_3 \times C_3) = 2$, then every crossing involves a red and a blue edge. Call e the red edge of one crossing and f the blue edge of the other crossing. Removing e and f leaves a plane drawing. Due to symmetry, there are only three possibilities for the relative positions of e and f. In two cases, $C_3 \times C_3 - \{e, f\}$ still contains a subdivision of $K_{3,3}$ as shown in FIGURE 7. This is impossible. For the third case, a result of Whitney **[22]** shows that any plane drawing of $C_3 \times C_3 - \{e, f\}$ must be essentially

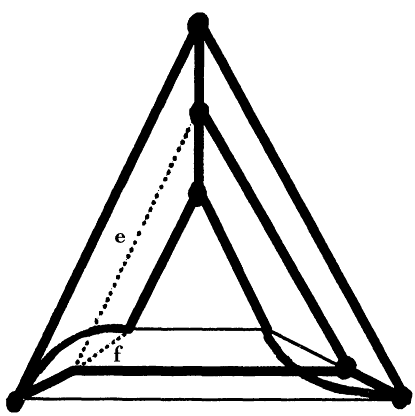

FIGURE 7
$K_{3,3}$ in $C_3 \times C_3 - \{e, f\}$.

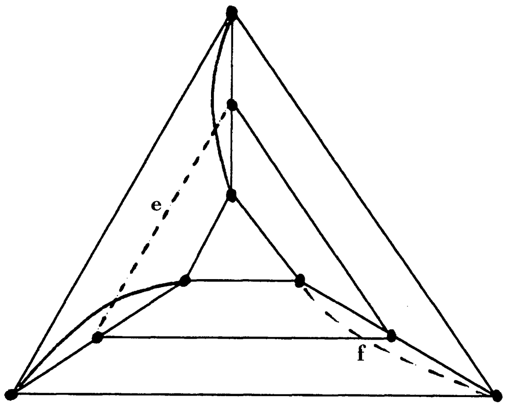

FIGURE 8
A plane drawing of $C_3 \times C_3 - \{e, f\}$.

that of FIGURE 8. But in that figure, e and f cannot be redrawn without producing at least three crossings. Thus there is no drawing of $C_3 \times C_3$ with only two crossings, and we conclude that $\nu(C_3 \times C_3) = 3$.

LEMMA 1. *For $n \geq 4$, let D be any drawing of $C_3 \times C_n$ in which no triangle has an edge crossed. Then D has at least n crossings.*

THEOREM $(3, n)$. $\nu(C_3 \times C_n) = n$.

Proof. We use Theorem $(3,3)$ as a base case for induction. Assume $\nu(C_3 \times C_k) = k$, and suppose there is a drawing of $C_3 \times C_{k+1}$ with fewer than $k+1$ crossings. By Lemma 1, some triangle has an edge crossed. Removing this triangle gives a drawing of $C_3 \times C_k$ with fewer than k crossings, thus contradicting the induction hypothesis. So every drawing of $C_3 \times C_{k+1}$ has at least $k+1$ crossings. Since FIGURE 1 shows that there is a drawing of $C_3 \times C_{k+1}$ with $k+1$ crossings, we conclude that $\nu(C_3 \times C_{k+1}) = k+1$. This completes the induction.

Proof summary for Lemma 1. Let D be any drawing of $C_3 \times C_n$ in which no triangle has an edge crossed. Color the 3-cycles red and the n-cycles blue. Define the *responsibility* of a subgraph H of a graph G to be the number of times the edges of H are crossed. Thus if one edge of H is crossed by an edge of G not in H, it contributes 1 to the responsibility of H, but if two edges of H cross, they contribute 2.

Let the subgraphs H_i of $C_3 \times C_n$ be triangular prisms consisting of two successive (red) triangles, T_i on $\{a_i, b_i, c_i\}$ and T_{i+1} on $\{a_{i+1}, b_{i+1}, c_{i+1}\}$, and their connecting (blue) edges $a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}$ (see FIGURE 9). Since no edge of a triangle is

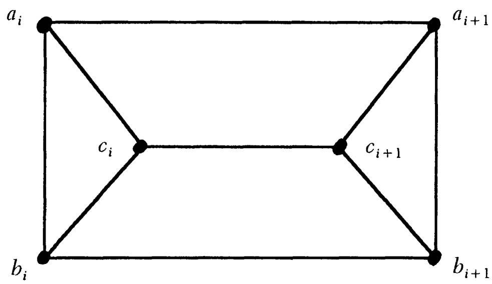

FIGURE 9
The subgraph H_i.

crossed, all crossings must involve blue edges. For a given H_i, there are two possibilities: Either H_i is planar, or H_i has two edges that cross. In the latter case H_i clearly has responsibility of at least 2. In the former the responsibility of H_i is due to edges outside H_i crossing edges of H_i. Because no triangle is crossed, T_{i+2} must lie in one of the quadrangular faces of H_i. Since blue n-cycles connect successive a_k (or b_k or c_k), an examination of cases shows that some blue cycle must cross H_i at least twice. Thus H_i has responsibility at least 2. This gives $C_3 \times C_n$ a total responsibility of at least $2n$. Since each blue edge is in exactly one H_i, D has at least n crossings.

7. The middle ground

Beineke and Ringeisen **[4]** soon went on to verify the $(4, n)$-conjecture. Using edge coloring, a particular situation, and induction, they showed that $\nu(C_4 \times C_n) = 2n$. Their induction used as a base case a result of Eggleton **[7]** showing that $\nu(C_4 \times C_4) = 8$. Here we outline a more recent proof, due to Dean and Richter **[5]**. The following lemma aids their argument.

LEMMA 2. *Every optimal drawing of $C_4 \times C_4$ contains a 4-cycle that is crossed at least four times.*

THEOREM (4,4). $\nu(C_4 \times C_4) = 8$.

Proof summary. Suppose that $\nu(C_4 \times C_4) < 8$ and that D is an optimal drawing of $C_4 \times C_4$. By Lemma 2, D has a 4-cycle that is crossed at least 4 times. Removing the 4-cycle leaves a drawing of $C_3 \times C_4$ with fewer than 4 crossings, contradicting Theorem $(3, n)$. Therefore $\nu(C_4 \times C_4) = 8$.

Any drawing of $C_m \times C_n$ has many 4-cycles. *Principal* 4-cycles are analogous to the 3-cycles or n-cycles in $C_3 \times C_n$. To verify the $(4, n)$-conjecture, Beineke and Ringeisen adapted their "responsibility" argument of Lemma 1 to prove the lemma below. Theorem $(4, n)$ follows by induction and is left to the reader.

LEMMA 3. *If D is a drawing of $C_4 \times C_n$ in which no principal 4-cycle has more than one crossing, then D has at least $2n$ crossings.*

THEOREM $(4, n)$. $\nu(C_4 \times C_n) = 2n$.

8. Recent results

The most recent results on the (m, n)-conjecture are due to Richter and Thomassen **[15]**, Kleŝc, Richter, and Stobert **[12]**, Anderson, Richter, and Rodney **[1, 2]**, and Salazar **[17]**.

In 1995 Richter and Thomassen published proofs of Theorem (4, 4) and Theorem (5, 5) using methods quite different from those we have reviewed so far. They began by introducing curve systems. A *curve system* comprises two families of curves, with the property that every curve in one family must intersect every curve in the other. By considering conditions of separation, disjointness, and optimality, they determined the smallest number of intersections that such curve systems can have.

Richter and Thomassen view a drawing of $C_m \times C_n$ as a curve system having one family of nm-cycles and another family of mn-cycles. Vertices provide the required intersections. Essentially, when $m = n = 4$, they prove that such a curve system has at least twenty-four intersections. Since $C_4 \times C_4$ has sixteen vertices, there remain at least eight crossings. Similarly, Richter and Thomassen show that a system of two families of five curves each has at least forty intersections. Thus any drawing of $C_5 \times C_5$ has at least fifteen crossings. This fact and FIGURE 1 verify the (5,5)-conjecture.

THEOREM (5,5). $\nu(C_5 \times C_5) = 15$.

In an undergraduate honors project guided by Richter, Stobert [**18**] proved the following extension. Since it was also proved independently by Klešc, the three produced a joint paper [**12**].

THEOREM (5, n). $\nu(C_5 \times C_n) = 3n$.

Proof summary. Klešc, Richter, and Stobert employ the now-familiar pattern of proof originated by Ringeisen and Beineke. Using red 5-cycles and blue n-cycles, they argue through several cases that if a drawing D of $C_5 \times C_n$ contains no red 5-cycle with more than two crossings, then D must have at least $3n$ crossings. If there is a 5-cycle with at least 3 crossings, they apply the induction assumption after deleting the 5-cycle.

Anderson, Richter, and Rodney continued the use of curve systems, and verified the (6,6)- and the (7,7)-conjecture. Salazar extended their results to $C_6 \times C_n$. Thus:

THEOREM (6, n). $\nu(C_6 \times C_n) = 4n$.

THEOREM (7,7). $\nu(C_7 \times C_7) = 35$.

Salazar also showed that, for an arbitrary integer M, the minimum number of crossings in any drawing of $C_m \times C_n$ in which no two n-cycles cross more than M times approaches $(m-2)n$ as n approaches infinity. "Thus in some sense," wrote Salazar in a letter to the author, "we can say that the [(m, n)-conjecture] is asymptotically true."

9. Conclusion

One wonders how much hope there is of establishing $\nu(C_m \times C_n)$ in general, given the slow progress made since the initial conjecture. As we have seen, for $\nu(C_m \times C_n)$, the m-values have resisted an inductive argument. What seems missing is some general method of relating $\nu(C_{m+1} \times C_{m+1})$ to $\nu(C_m \times C_m)$ or to $\nu(C_m \times C_{m+1})$. On the other hand, it may be that the question of the crossing number of $C_m \times C_n$ is simply one that "the wisest [person] cannot answer."

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Unity in a Field

I was running through a field,
On a normal, warm spring day,
Though it seemed a bit improper,
I felt free in my own way.

This independent feeling—
A direct product of my mind,
Of which the field's an extension,
Had a basis I could find.

The field was not complex.
Though it seemed a bit unreal.
I thought it spanned forever.
But it's finite, I now feel.

I saw a ring of operators,
Radicals, no doubt.
They were planning a group action.
And quickly closed me out.

So I formed a group myself.
My identity sufficed.
You may think that it was trivial.
But I thought it was nice.

Some say it would be ideal,
Were I to find a friend.
But here there are just two ideals,
The Field and me. The End.

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Unity in a Field

I was running through a field,
On a normal, warm spring day,
Though it seemed a bit improper,
I felt free in my own way.

This independent feeling—
A direct product of my mind,
Of which the field's an extension,
Had a basis I could find.

The field was not complex.
Though it seemed a bit unreal.
I thought it spanned forever.
But it's finite, I now feel.

I saw a ring of operators,
Radicals, no doubt.
They were planning a group action.
And quickly closed me out.

So I formed a group myself.
My identity sufficed.
You may think that it was trivial.
But I thought it was nice.

Some say it would be ideal,
Were I to find a friend.
But here there are just two ideals,
The Field and me. The End.

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Leibniz's Formula, Cauchy Majorants, and Linear Differential Equations

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1. Introduction

Students of differential equations are familiar with power series solutions of second-order linear differential equations with variable coefficients, written

$$a(t)y''(t)+b(t)y'(t)+c(t)y(t)=f(t). \tag{1}$$

The standard textbook method suggests that one assume the existence of a power series solution, substitute this into the equation, rearrange the relevant power series, and then equate coefficients of like powers; this yields recurrence relations for the power series coefficients.

We will outline a method, which goes back to Cauchy, augmented by our use of the Leibnitz formula for product differentiation. Combining these two ideas provides a method that is vastly more efficient—both pedagogically and from the standpoint of symbolic algebra manipulations. In particular, one can write down the recurrence relation by inspection, without any computations. A complete discussion of this and related issues appears in our book **[3]**. In the final section we briefly discuss the history of Cauchy's method.

2. Essentials of the method

First we outline the method for the general second-order linear ordinary differential equation (1). We assume that the coefficient functions a, b, c, f have convergent power series expansions around the base point $t=t_0$, and that $a(t_0)\neq 0$. We search for a power series solution

$$y(t)=\sum_{n=0}^{\infty}Y_n(t-t_0)^n=\sum_{n=0}^{\infty}\frac{y_n}{n!}(t-t_0)^n.$$

The values of y and its successive derivatives at the base point are given by

$$y(t_0)=Y_0=y_0,\ y'(t_0)=Y_1=y_1,\ y''(t_0)=2Y_2=y_2,\ \ldots,\ y^{(n)}(t_0)=n!Y_n=y_n,$$

and similarly for a, b, c, f. To simplify the formulas below, it is convenient to work with the y_n coefficients. Thus we write

$$a(t)=\sum_{n=0}^{\infty}\frac{a_n}{n!}(t-t_0)^n,\quad b(t)=\sum_{n=0}^{\infty}\frac{b_n}{n!}(t-t_0)^n,$$

$$c(t)=\sum_{n=0}^{\infty}\frac{c_n}{n!}(t-t_0)^n,\quad\text{and}\quad f(t)=\sum_{n=0}^{\infty}\frac{f_n}{n!}(t-t_0)^n,$$

where $a_n=a^{(n)}(t_0)$, and similarly for b, c, and f.

We also need Leibniz's binomial formula for product differentiation ([8], p. 540)

$$(AB)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} A^{(k)} B^{(n-k)}. \tag{2}$$

When $n = 1$ and $n = 2$ this specializes to the familiar Leibniz rules

$$(AB)' = AB' + A'B; \quad (AB)'' = AB'' + 2A'B' + A''B.$$

If $y(t)$ is a solution of the differential equation (1) with given initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$, then substitution in (1) at the base point reveals

$$a_0 y_2 + b_0 y_1 + c_0 y_0 = f_0; \tag{3}$$

this gives the coefficient y_2 in terms of $\{a_0, b_0, c_0, y_0, y_1, f_0\}$. Differentiating (1) at the base point gives

$$(a_0 y_3 + a_1 y_2) + (b_0 y_2 + b_1 y_1) + (c_0 y_1 + c_1 y_0) = f_1, \tag{4}$$

which can be solved for y_3 in terms of y_2 (the solution to equation (3)), and $\{a_0, a_1, b_0, b_1, c_0, c_1, y_0, y_1, f_1\}$. To obtain a general formula for y_n, we apply the binomial derivative formula (2) to (1) to obtain, for $n = 0, 1, \dots,$

$$\boxed{a_0 y_{n+2} + \sum_{k=1}^{n} \binom{n}{k} a_k y_{n+2-k} + \sum_{k=0}^{n} \binom{n}{k} (b_k y_{n+1-k} + c_k y_{n-k}) = f_n,} \tag{5}$$

where we have isolated the first term involving y_{n+2}.

This is the general form of the recurrence relation. We required no re-labeling of sums or "index-shifting." It is especially easy to remember if we note the descending levels of homogeneity of the indices: the terms involving a_k have indices totalling $n + 2$, the terms involving b_k have indices totalling $n + 1$, and the terms involving c_k have indices totalling n.

EXAMPLE. We illustrate with the Airy equation $y'' - ty = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$. Here we have $b(t) \equiv 0$ and $c(t) = -t$, so $b_k \equiv 0$, $c_1 = -1$ and $c_k = 0$ otherwise, and $y_0 = 0$, $y_1 = 1$. The recurrence relation (5) gives

$$y_2 = 0 \quad \text{and} \quad y_{n+2} - n y_{n-1} = 0 \quad \text{for } n \geq 1.$$

It follows easily that $0 = y_0 = y_2 = y_3 = y_5 = y_6 = y_8 = \cdots = y_{3n} = y_{3n+2}$, and that $y_4 = 2y_1 = 2$; $y_7 = 5y_4 = 5 \times 2$. In general,

$$y_{3n+1} = 2 \times 5 \times \cdots \times (3n - 1),$$

which gives the required solution

$$y(t) = t + \frac{2t^4}{4!} + \frac{10t^7}{7!} + \cdots + \frac{2 \times 5 \times \cdots \times (3n-1) t^{3n+1}}{(3n+1)!} + \cdots.$$

3. Proof of convergence

It is not difficult to prove the convergence of the power series solution obtained by this method. Cauchy's "method of majorants" [6] works as follows: replace the coefficients $a(t), b(t), c(t), f(t)$ by power series for which we *can* solve the equation, and which serve as suitable upper bounds for these coefficients. Since any power series that converges for $|t-t_0|<R$ is dominated by a suitable geometric series, it is natural to consider majorants of the form $\bar{b}(t)=(1-(t-t_0)/R)^{-1}$ and closely related functions. On the other hand, the function $Y(t)=(1-(t-t_0)/R)^{-r}$ is easily seen to be the solution of a Cauchy-Euler equation of the form

$$Y''-\frac{BY'}{(1-(t-t_0)/R)}-\frac{CY}{(1-(t-t_0)/R)^2}=0$$

for suitable values of B, C, R, and r.

Since $a(t_0)\neq 0$, we have $a(t)\neq 0$ in some interval about t_0, so we can divide equation (1) by $a(t)$ and restrict attention to equations of the form

$$y''+b(t)y'(t)+c(t)y(t)=f(t). \tag{6}$$

The homogeneous equation First we consider the homogenous equation

$$y''+b(t)y'+c(t)y=0. \tag{7}$$

A further simplification is to replace $(t-t_0)/R$ by t, which is equivalent to assuming that $t_0=0$ and that the functions $b(t)$, $c(t)$ have power series that converge for $|t|\leq 1$. The problem, then, is to show that the power series solution converges for $|t|<1$.

Since the given power series are convergent at $t=1$, the coefficients must be bounded:

$$\frac{|b_n|}{n!}\leq B,\quad \frac{|c_n|}{n!}\leq C,\quad n=0,1,2,\dots,$$

where B and C are positive numbers. We now consider the related functions

$$\frac{B}{1-t}=\sum_{n=0}^{\infty} Bt^n=\sum_{n=0}^{\infty}\frac{\bar{b}_n}{n!}t^n$$

and

$$\frac{C}{(1-t)^2}=\sum_{n=0}^{\infty} C(n+1)t^n=\sum_{n=0}^{\infty}\frac{\bar{c}_n}{n!}t^n,$$

where

$$|b_n|\leq Bn!=\bar{b}_n \quad\text{and}\quad |c_n|\leq C(n+1)!=\bar{c}_n. \tag{8}$$

We now consider the initial-value problem for the Cauchy-Euler equation

$$Y''(t)-\frac{B}{1-t}Y'(t)-\frac{C}{(1-t)^2}Y(t)=0,\quad Y(t_0)=\bar{y}_0,\quad Y'(t_0)=\bar{y}_1.$$

As mentioned above, a trial solution is sought in the form $(1-t)^{-r}$. Since this equation is linear and homogeneous it is natural to look for the general solution in the form

$$Y(t)=\alpha_1(1-t)^{-r_1}+\alpha_2(1-t)^{-r_2}=\sum_{n=0}^{\infty}\frac{\bar{y}_n}{n!}t^n, \tag{9}$$

where α_1, α_2 are constants and r_1 and r_2 are the roots of the quadratic equation $r(r+1) - Br - C = 0$. This quadratic has two real roots $r_1 < 0 < r_2$, where

$$2r_{1,2} = (B-1) \pm \sqrt{(B-1)^2 + 4C}\,.$$

Using the initial conditions $Y(0) = \bar{y}_0$ and $Y'(0) = \bar{y}_1$, we can solve for α_1, α_2 to obtain the required solution $Y(t)$.

From the recurrence relation of equation (5), we must have

$$\bar{y}_{n+2} - \sum_{k=0}^{n} \binom{n}{k}\left(\bar{b}_k \bar{y}_{n+1-k} + \bar{c}_k \bar{y}_{n-k}\right) = 0, \quad n = 0,1,2,\dots. \tag{10}$$

This shows the important fact that if $\bar{y}_0 \geq 0$ and $\bar{y}_1 \geq 0$, then $\bar{y}_n \geq 0$ for all n.

To obtain two linearly independent solutions of (7), we first take $(y_0, y_1) = (1,0)$, then $(y_0, y_1) = (0,1)$; in each case we use the same values for $(\bar{y}_0, \bar{y}_1)$. We propose to show that $|y_n| \leq \bar{y}_n$ for all n. This is clearly satisfied for $n = 0,1$ by definition. Assuming that the inequality holds for the indices $2, \dots, n+1$, we conclude from (5), (8), (10) that

$$\begin{aligned} |y_{n+2}| &= \left| \sum_{k=0}^{n} \binom{n}{k}(b_k y_{n+1-k} + c_k y_{n-k}) \right| \\ &\leq \sum_{k=0}^{n} \binom{n}{k}\left(\bar{b}_k \bar{y}_{n+1-k} + \bar{c}_k \bar{y}_{n-k}\right) \\ &= \bar{y}_{n+2}\,. \end{aligned}$$

Thus the coefficients y_n are majorized by the coefficients of a convergent power series, so the series $\sum y_n t^n/n!$ is convergent, as desired.

The inhomogeneous equation To complete the proof, it suffices to find a particular solution to the inhomogeneous equation, since the general solution of (6) is the sum of a particular solution together with the general solution of the homogeneous equation (7), just constructed.

Again, we relocate the origin and rescale so that $t_0 = 0$ and the power series for $b(t)$, $c(t)$, and $f(t)$ converge for $|t| \leq 1$. In particular, all the coefficients must be bounded at $t = 1$:

$$\frac{|b_n|}{n!} \leq B, \quad \frac{|c_n|}{n!} \leq C, \quad \frac{|f_n|}{n!} \leq F,$$

for positive constants B, C, and F. Now examine the polynomial

$$p(r) = r(r+1) - Br - C.$$

Let $r = \left((B-1) + \sqrt{(B-1)^2 + 4(C+F)}\right)/2$; note that $r > 0$ and $p(r) = F$. Hence we have the bounds

$$|b_n| \leq Bn! = \bar{b}_n, \quad |c_n| \leq C(n+1)! = \bar{c}_n, \quad |f_n| \leq \frac{F(n+r+1)!}{(r+1)!} = \bar{f}_n.$$

Now consider the function

$$Y(t) = (1-t)^{-r} = \sum_{n=0}^{\infty} \frac{\bar{y}_n}{n!} t^n.$$

Direct computation shows that $Y(t)$ satisfies the equation

$$Y'' - \frac{B}{(1-t)}Y'(t) - \frac{C}{(1-t)^2}Y(t) = \frac{F}{(1-t)^{-r-2}} = \sum_{n=0}^{\infty} \frac{\bar{f}_n}{n!}t^n.$$

Hence the Taylor coefficients satisfy

$$0 \le \bar{y}_{n+2} = \sum_{k=0}^{n} \binom{n}{k}\left(\bar{b}_k \bar{y}_{n+1-k} + \bar{c}_k \bar{y}_{n-k}\right) + \bar{f}_n.$$

We obtain a particular solution by taking $y_0 = 1 = \bar{y}_0$ and $y_1 = r = \bar{y}_1$ and define the higher Taylor coefficients by solving the recurrence relation (5). The Taylor coefficients $\bar{y}_n$ of the majorant are all non-negative; we propose to show by induction that $|y_n| \le \bar{y}_n$ for all n. This is true for $n = 0,1$ by definition; assuming the truth for $y_2, \dots y_{n+1}$, we have

$$\begin{aligned} |y_{n+2}| &\le |f_n| + \left| \sum_{k=0}^{n} \binom{n}{k}(b_k y_{n+1-k} + c_k y_{n-k}) \right| \\ &\le \bar{f}_n + \sum_{k=0}^{n} \binom{n}{k}\left(\bar{b}_k \bar{y}_{n+1-k} + \bar{c}_k \bar{y}_{n-k}\right) \\ &= \bar{y}_{n+2}; \end{aligned}$$

this completes the proof. We have found a particular power series solution of the differential equation (6).

4. Extension to N^{th}-order equations

To extend the previous method to general N^{th}-order equations, we first introduce a convenient notation. The general N^{th}-order linear differential equation is written

$$p_N(t)\,y^{(N)}(t) + p_{N-1}(t)\,y^{(N-1)}(t) + \cdots + p_0(t)\,y(t) = f(t), \tag{11}$$

where $p_N(t_0) \neq 0$. If we define

$$\begin{aligned} \mathbf{P} &= (p_N(t), p_{N-1}(t), \dots, p_0(t)) \\ \mathbf{Y} &= (y^{(N)}(t), y^{(N-1)}(t), \dots, y(t)), \end{aligned}$$

then equation (11) can be written

$$\mathbf{P} \cdot \mathbf{Y} = f(t). \tag{12}$$

Again, the general form of the n^{th}-order recurrence relation requires that we repeatedly differentiate equation (12) at the base point. Note that

$$\begin{aligned} (\mathbf{P}\cdot\mathbf{Y})' &= \mathbf{P}\cdot\mathbf{Y}' + \mathbf{P}'\cdot\mathbf{Y} \quad \text{and} \\ (\mathbf{P}\cdot\mathbf{Y})'' &= \mathbf{P}\cdot\mathbf{Y}'' + 2\mathbf{P}'\cdot\mathbf{Y}' + \mathbf{P}''\cdot\mathbf{Y}. \end{aligned}$$

More generally, Leibniz's binomial derivative formula gives

$$(\mathbf{P}\cdot\mathbf{Y})^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \mathbf{P}^{(k)} \cdot \mathbf{Y}^{(n-k)}.$$

This formula is identical to the original Leibniz formula for product differentiation where ordinary multiplication is replaced by the dot product.

If $f(t)$ and $\mathbf{P}(t)$ have power series expansions around the base point $t = t_0$, then we may write

$$f(t) = \sum_{n=0}^{\infty} \frac{f_n}{n!}(t-t_0)^n \quad \text{and} \quad p_k(t) = \sum_{n=0}^{\infty} \frac{p_{(k,n)}}{n!}(t-t_0)^n \quad \text{for} \quad k = 0, 1, \ldots, N,$$

and search for a power series solution of the form

$$y(t) = \sum_{n=0}^{\infty} \frac{y_n}{n!}(t-t_0)^n.$$

If $y(t)$ is a solution to equation (11) with initial conditions $y(t_0) = y_0$, $y'(t_0) = y_1, \ldots, y^{(N-1)}(t_0) = y_{N-1}$, substitution into equation (11) at the base point yields

$$(\mathbf{P} \cdot \mathbf{Y} = f(t))\big|_{t \to t_0} \Rightarrow p_{(N,0)}\, y_N + p_{(N-1,0)}\, y_{N-1} + \cdots + p_{(0,0)}\, y_0 = f_0.$$

This can be solved for y_N in terms of $\{p_{(N,0)}, p_{(N-1,0)}, \ldots, p_{(0,0)}, f_0\}$ and the initial conditions $\{y_{N-1}, y_{N-2}, \ldots, y_0\}$. This is precisely the extension of equation (3) to the N^{th}-order equation. If we differentiate equation (12) n times and apply the binomial formula, we obtain

$$\left((\mathbf{P} \cdot \mathbf{Y})^{(n)} = f^{(n)}(t)\right)\Big|_{t \to t_0} \Rightarrow \sum_{k=0}^{n} \binom{n}{k} \mathbf{P}_k \cdot \mathbf{Y}_{n-k} = f_n. \tag{13}$$

We may also rewrite equation (13) by isolating the only term involving y_{n+N}, and then solving for y_{n+N} to obtain the desired solution. This yields a form that extends equation (5) to the N^{th}-order case: for $n = 0, 1, \ldots,$

$$\boxed{p_{(N,0)}\, y_{n+N} + \sum_{k=1}^{N} p_{(N-k,0)}\, y_{n+N-k} + \sum_{k=1}^{n} \binom{n}{k} \mathbf{P}_k \cdot \mathbf{Y}_{n-k} = f_n.} \tag{14}$$

5. Extension to singular equations

The methods discussed above are easy to extend to second-order equations with a regular singular point, such as the Bessel equation.

Reformulation of the method The most general second-order homogeneous equation with a regular singular point at $t = t_0$ is written

$$(t-t_0)^2 y'' + (t-t_0)\,p(t)\,y' + q(t)\,y = 0,$$

where $p(t)$ and $q(t)$ are assumed to have power series expansions

$$p(t) = \sum_{n=0}^{\infty} \frac{p_n}{n!}(t-t_0)^n, \quad q(t) = \sum_{n=0}^{\infty} \frac{q_n}{n!}(t-t_0)^n,$$

which are convergent in some interval $|t - t_0| \le R$.

From the work of Frobenius ([1], Chapter 4), it is known that there exists a power series solution, of the form

$$y(t) = |t-t_0|^r \sum_{n=0}^{\infty} \frac{y_n}{n!}(t-t_0)^n,$$

where r is the larger root of the indicial equation

$$r(r-1) + p_0 r + q_0 = 0.$$

As in the previous discussion, it is no loss of generality to take $t_0 = 0$ in what follows. In the subsequent treatment we illustrate Cauchy's method in case the larger root is $r = 0$. The reduction from the general case to this case is carried out in complete detail in our book ([3], Chapter 21).

Since $r = 0$ is the larger root, we must have

$$q_0 = 0, \quad p_0 \geq 1.$$

To find the coefficients y_n we apply the product rule to each term in the equation:

$$\left[t^2 y''\right]^{(n)}\Big|_{t=0} = \left(t^2 y^{(n+2)} + 2nty^{(n+1)} + n(n-1)\,y^{(n)}\right)\Big|_{t=0} = n(n-1)\,y_n,$$

$$\left[tp(t)\,y'\right]^{(n)}\Big|_{t=0} = \sum_{k=1}^{n} \binom{n}{k} kp_{k-1}\,y_{n+1-k},$$

$$\left[q(t)\,y\right]^{(n)}\Big|_{t=0} = \sum_{k=1}^{n} \binom{n}{k} q_k\,y_{n-k}.$$

(We used the easily-proved fact that $[tp(t)]^{(k)}|_{t=0} = kp_{k-1}$.) This yields the recurrence relation in the form

$$\boxed{n(n-1)\,y_n + \sum_{k=1}^{n}\binom{n}{k} kp_{k-1}\,y_{n+1-k} + \sum_{k=1}^{n}\binom{n}{k} q_k\,y_{n-k} = 0.} \qquad (15)$$

In particular, for $n = 1, 2, 3$:

$$p_0 y_1 + q_1 y_0 = 0,$$
$$(2+2p_0)\,y_2 + (2p_1 + 2q_1)\,y_1 + q_2 y_0 = 0,$$
$$(6+3p_0)\,y_3 + (6p_1 + 3q_1)\,y_2 + (3p_2 + 3q_2)\,y_1 + q_3 y_0 = 0.$$

We note that, in contrast with the power series solution about an ordinary point, the sum of the indices in each term in the recurrence relation is constant. We note also that the higher coefficients are uniquely determined by y_0; indeed, since $p_0 \geq 1$, we see that the coefficient of y_n is $n(n-1) + np_0 \geq n^2 \neq 0$ for $n \geq 1$.

EXAMPLE. We illustrate by finding a power series solution of Bessel's equation of order zero: $t^2y'' + ty' + t^2y = 0$. Here $p(t) = 1$ and $q(t) = t^2$, so $p_0 = 1$, $q_2 = 2$ and all other coefficients are zero. The recurrence relation for y_n reduces to $y_1 = 0$ and, for $n \geq 2$,

$$n^2 y_n + n(n-1)\,y_{n-2} = 0.$$

Thus $y_n = 0$ for odd n; for even n we have

$$y_{2m} = \frac{-(2m-1)}{2m} \cdots \frac{-3}{4}\frac{-1}{2} y_0 = (-1)^m \frac{(2m)!}{[2^m m!]^2} y_0,$$

which gives the well-known power series for $J_0(t)$ when $y_0 = 1$.

Proof of convergence Convergence can be proved directly, beginning with equation (15). We assume that $R = 1$ so that the coefficients are bounded by a constant $A > 1$:

$$\left|\frac{p_k}{k!}\right| \le A, \quad \left|\frac{q_k}{k!}\right| \le A, \quad k = 0, 1, 2, \dots .$$

We set $Y_k = y_k / k!$, so that from equation (15) we have

$$n^2 |y_n| \le (n(n-1) + np_0)|y_n| \le n! \left(A \sum_{k=1}^{n} |Y_{n+1-k}|(n+1-k) + A \sum_{k=1}^{n} |Y_{n-k}| \right).$$

Dividing both sides by $n^2 n!$ results in the inequality

$$|Y_n| \le \frac{2A}{n} \sum_{k=0}^{n-1} |Y_k|.$$

We have $Y_0 = 1$ and $Y_1 = |q_1/p_0| \le A$. We show by mathematical induction that for all k,

$$|Y_k| \le Ak^{\alpha} \quad \text{where} \quad \alpha = 2A - 1.$$

Indeed, this is true for $k = 1$; assuming the truth for $k = 1, \dots, n-1$ we have

$$|Y_n| \le \frac{2A}{n} \sum_{k=0}^{n-1} |Y_k| \le \frac{2A}{n} \sum_{k=0}^{n-1} Ak^{\alpha} \le \frac{2A}{n} \frac{An^{\alpha+1}}{\alpha+1} = An^{\alpha}.$$

Therefore, the power series $\sum_n Y_n t^n$ converges for $|t| < 1$, which was to be proved.

6. Historical notes

Cauchy's original paper was published on July 4, 1842, under the title *Memoire sur l'emploi du nouveau calcul, appellé 'calcul des limites' dans l'intégration des equations differentielles* [4]. In this work it is first noted that any single N^{th}-order differential equation, in general non-linear, can be written as a system of first-order equations. Then it is shown that if all of the defining functions are analytic in the neighborhood of the base point, the system may be differentiated successively at the base point to find the Taylor coefficients of the unknown solution. The *'calcul des limites'* refers to finding a suitable majorant equation in order to prove convergence of the resultant power series.

Cauchy's method was well exposited in textbooks in the earlier part of the twentieth century, especially in the very popular book of Goursat [7]. The method seems to have been largely forgotten in the post-war period, with the exception of [2]. For the specific case of second-order linear equations, the recurrence formula is derived in

[5]. More recently Strichartz [6] gave this derivation together with the explicit construction of majorants for linear second-order equations, as we have done. None of these works uses the streamlined formulation possible with the binomial formula.

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NOTES

Will the Real Non-Euclidean Parabola Please Stand Up?

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The parabola has two outstanding geometric properties: the *locus property* (it is the locus of points equidistant from a point and a line) and the *reflecting property* (it reflects parallel lines to a single point). While the locus property is usually considered the defining property of the parabola, the reflecting property also distinguishes parabolas among all other plane curves, and hence could be taken as its definition in the Euclidean plane.

Surprisingly, however, the curves satisfying these properties in the hyperbolic (Lobachevskian) plane are *different*. This raises the question: What is the *real* non-Euclidean parabola: the curve satisfying the locus property, or the curve satisfying the reflecting property?

The purpose of this note is to derive equations for both these curves, and address this question. This extends the discussion of non-Euclidean parabolas begun by Ron Perline [**2**]. Perline used Poincaré's half-plane model. Our treatment uses homogeneous coordinates.

Homogeneous coordinates A point **P** in the hyperbolic plane has three homogeneous, real coordinates

$$\mathbf{P} = (x, y, z) \tag{1}$$

where $z^2 - x^2 - y^2 > 0$ and we usually assume also $z > 0$. These points are in the interior of a cone in three-dimensional Euclidean space, the cone, $\mathscr{C}$, with equation $z^2 - x^2 - y^2 = 0$ (see FIGURE 1). As usual with homogeneous coordinates, any non-zero scalar multiple of the coordinates (1) represents the *same* point in the hyperbolic plane, i.e., **P** also has coordinates (kx, ky, kz) for any $k \neq 0$. When we need *unique* coordinates for **P**, we normalize by choosing $k = (z^2 - x^2 - y^2)^{-1/2}$ which puts **P** on the upper sheet of the hyperboloid of two sheets, $\mathscr{H}$, with equation: $z^2 - x^2 - y^2 = 1$.

Let us call $\mathscr{H}$ the **hyperboloid model**. This is the model of hyperbolic geometry that is closely linked to the theory of relativity. The hyperboloid $\mathscr{H}$ is the "unit sphere" of three-dimensional Minkowski space. In turn, $\mathscr{H}$ is connected with the disk model by stereographic projection.

These concepts are depicted in FIGURE 1. The cone, $\mathscr{C}$, is a wire frame; the hyperboloid, $\mathscr{H}$, is gray. The thick line, determined by the point **P** and the origin, represents a single point in the hyperbolic plane. The normalized representative of this point is the point **P**. Stereographic projection from the point **S** with Cartesian coordinates $(0, 0, -1)$, is depicted by a thin line, and maps **P**, on the hyperboloid, to **Q**, inside the unit disk in the (x, y)-plane.

Straight lines Although less familiar than the disk and half-plane models of hyperbolic geometry, the hyperboloid model possesses a number of advantages. In the

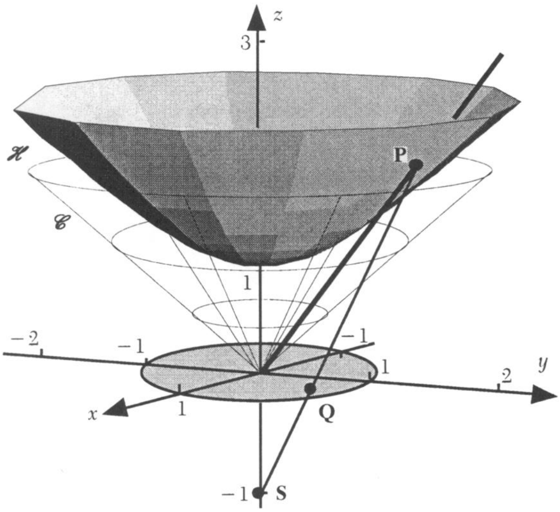

FIGURE 1
The hyperboloid model of hyperbolic geometry.

first place, the equation of a straight line is linear. If the coefficients a, b, c satisfy $c^2 - a^2 - b^2 < 0$ (and we usually assume $c > 0$ also), then the set of solutions (1) of the equation

$$ax + by + cz = 0 \tag{2}$$

is a straight line in the hyperbolic plane. We take $\mathbf{L} = [a, b, c]$ as homogeneous line coordinates. As with point coordinates, any non-zero scalar multiple $[ka, kb, kc]$ also serves as coordinates for $\mathbf{L}$. When we need unique coordinates, we choose $k = (a^2 + b^2 - c^2)^{-1/2}$. Regarded as a vector with its tail at the origin, the normalized $\mathbf{L}$ has its head on the hyperboloid of one sheet with equation: $z^2 - x^2 - y^2 = -1$, which we call $\mathscr{H}^*$. Normalized or not, the vector $\mathbf{L}$ is normal (i.e., perpendicular) to a plane through the origin whose intersection with $\mathscr{H}$ is the hyperbolic straight line of equation (2).

All this can be observed in FIGURE 2. The wire frame is the hyperboloid of one sheet $\mathscr{H}^*$; the hyperboloid of two sheets, $\mathscr{H}$, is still gray. The vector $\mathbf{L}$ determines a

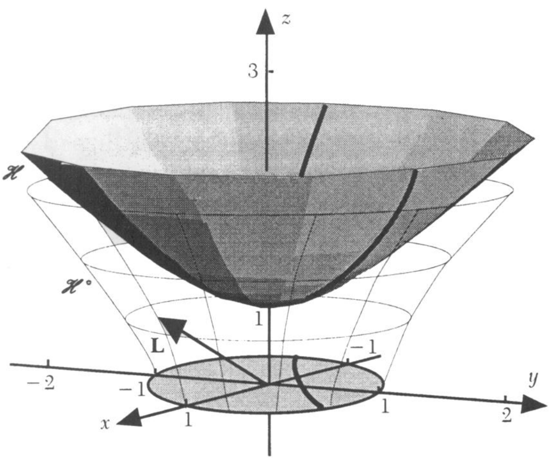

FIGURE 2
A straight line in the hyperbolic plane.

hyperbolic straight line, depicted as a thick curve on $\mathscr{H}$. The stereographic projection of this line is the thick curve in the (x, y)-plane: a circular arc perpendicular to the unit circle.

Mensuration formulae We now list the elegant formulae used to calculate distances and angles in homogeneous coordinates. Let $\mathbf{P} = (x_1, y_1, z_1)$ and $\mathbf{Q} = (x_2, y_2, z_2)$ be two points. We need two scalar products: the usual **dot product**

$$\mathbf{P} \cdot \mathbf{Q} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

and the **Minkowski product**

$$\langle \mathbf{P}, \mathbf{Q} \rangle = z_1 z_2 - x_1 x_2 - y_1 y_2,$$

plus the **norm**

$$\|\mathbf{P}\| = |\langle \mathbf{P}, \mathbf{P} \rangle|^{1/2}.$$

(A) *Given two points*: $\mathbf{P} = (x_1, y_1, z_1)$ and $\mathbf{Q} = (x_2, y_2, z_2)$, the line $\mathbf{L}$ determined by $\mathbf{P}$ and $\mathbf{Q}$ has homogeneous coordinates

$$\mathbf{L} = \mathbf{P} \times \mathbf{Q} = [y_1 z_2 - y_2 z_1, -x_1 z_2 + x_2 z_1, x_1 y_2 - x_2 y_1] \tag{3}$$

and the distance, $d(\mathbf{P}, \mathbf{Q})$, from $\mathbf{P}$ to $\mathbf{Q}$ is

$$d(\mathbf{P}, \mathbf{Q}) = \operatorname{arccosh}\left(\left| \frac{\langle \mathbf{P}, \mathbf{Q} \rangle}{\|\mathbf{P}\| \|\mathbf{Q}\|} \right| \right). \tag{4}$$

(B) *Given a point and a line*: $\mathbf{P} = (x, y, z)$ and $\mathbf{L} = [a, b, c]$, the perpendicular from $\mathbf{P}$ to $\mathbf{L}$ is the line, $\mathbf{M}$, with coordinates

$$\mathbf{M} = [cy + bz, -(az + cx), ay - bx] \tag{5}$$

and the distance from $\mathbf{P}$ to $\mathbf{L}$ is

$$d(\mathbf{P}, \mathbf{L}) = \operatorname{arcsinh}\left(\left| \frac{\mathbf{P} \cdot \mathbf{L}}{\|\mathbf{P}\| \|\mathbf{L}\|} \right| \right). \tag{6}$$

(C) *Given two lines*: $\mathbf{L} = [a_1, b_1, c_1]$ and $\mathbf{M} = [a_2, b_2, c_2]$, let

$$k = \left| \frac{\langle \mathbf{L}, \mathbf{M} \rangle}{\|\mathbf{L}\| \|\mathbf{M}\|} \right|.$$

Then $\mathbf{L}$ and $\mathbf{M}$ *intersect*, are *parallel*, or are *hyperparallel* according as $k < 1$, $k = 1$, or $k > 1$. If $\mathbf{L}$ and $\mathbf{M}$ intersect, the point of intersection is $\mathbf{P} = \mathbf{L} \times \mathbf{M}$, and the angle of intersection is

$$\theta = \arccos(k) = \arccos\left(\frac{|\langle \mathbf{L}, \mathbf{M} \rangle|}{\|\mathbf{L}\| \|\mathbf{M}\|} \right). \tag{7}$$

If $\mathbf{L}$ and $\mathbf{M}$ are hyperparallel, then their unique common perpendicular, called the **axis** of $\mathbf{L}$ and $\mathbf{M}$, is the line with coordinates

$$\mathbf{K} = [-b_1 c_2 + b_2 c_1, a_1 c_2 - a_2 c_1, a_1 b_2 - a_2 b_1] \tag{8}$$

while the shortest distance between $\mathbf{L}$ and $\mathbf{M}$ is $s = \operatorname{arccosh}(k)$.

These formulas are derived in Coxeter [1]. We won't need all of them (but to leave even one out spoils their elegance as a group). Note that (5) and (8) are not cross products (but do have some cross product-like properties). Finally, although we use homogeneous coordinates for calculation, we draw figures in the disk model (where straight lines are arcs of circles perpendicular to the unit circle) because this gives such a good overall view of the hyperbolic plane.

The locus property Armed with these formulas, let's find the equation of the curve in the hyperbolic plane that is the locus of points equidistant from a given point **F** and line **L**. For convenience we put **F** on the x-axis a unit distance from the origin. The line **L** we also place a unit distance from the origin, perpendicular to the x-axis. (See FIGURE 3.) The result is that the origin, being equidistant from **F** and **L**, is on the locus, in fact, is the vertex of our "parabola."

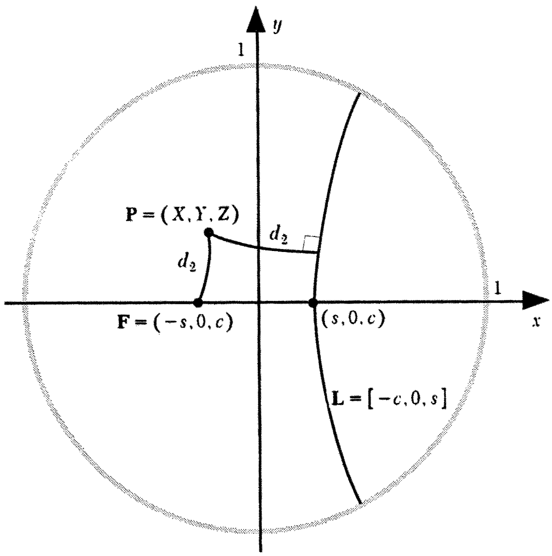

FIGURE 3
Points **P** such that $d_1 = d_2$ form a parabolic locus.

We need coordinates for **F** and **L**. Since the x-axis has coordinates $[0,1,0]$ (its equation is $y = 0$), **F** has coordinates $(x, 0, z)$ for some x and z which we assume, for simplicity, are chosen so that **F** is normalized: $z^2 - x^2 = 1$. The distance from **F** to the origin $(0,0,1)$ is 1, so the distance formula (4) gives $z = \cosh(1)$. Therefore, $x = -\sinh(1)$ and **F** has coordinates $(-s, 0, c)$ where $s = \sinh(1)$ and $c = \cosh(1)$. Next, since **L** is the perpendicular to the x-axis at the point $(s, 0, c)$, the point symmetric to **F** on the other side of the origin, $\mathbf{L} = [-c, 0, s]$ by formula (5).

Let $\mathbf{P} = (X, Y, Z)$ represent any point on the locus. We assume that **P** is normalized. Then $d_1 = \operatorname{arccosh}(|\langle \mathbf{P}, \mathbf{F} \rangle|)$ according to formula (4). On the other hand, $d_2 = \operatorname{arcsinh}(|\mathbf{P} \cdot \mathbf{L}|)$ by formula (6). Setting these equal,

$$|cZ + sX| = \sqrt{1 + (sZ - cX)^2} \tag{9}$$

and solving together with the normalization condition: $Z^2 - X^2 - Y^2 = 1$, gives the equation

$$Y^2 = -4csXZ, \tag{10}$$

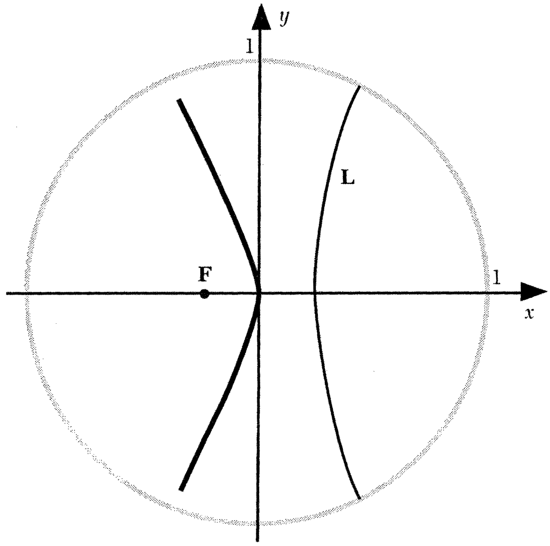

FIGURE 4
The locus of points equidistant from **F** and **L**.

whose graph, in the disk model of the hyperbolic plane, is given in FIGURE 4. Equation (10) is preferable to equation (9) not only because it is simpler, but because, being homogeneous, it is satisfied by *all* coordinates of points on the locus, *not* just normalized coordinates.

FIGURE 4 looks quite like a Euclidean parabola, although perhaps a bit too bent at the "vertex." Equation (10) is also very like the equation of the Euclidean parabola under analogous circumstances: $y^2 = -4x$. We return later to the question of the genuineness of this parabola.

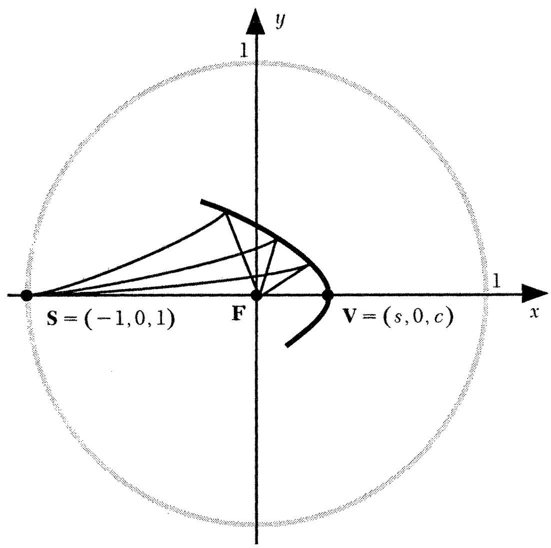

FIGURE 5
The reflecting property in the hyperbolic plane.

The reflecting property We turn to the parabolic reflector: a curve that reflects parallel rays (coming from a given ideal point) to a given ordinary point. We place the source of the light rays, **S**, on the negative x-axis and the focus, **F**, at the origin. Our goal is to find the equation of the "parabola" passing through the point $\mathbf{V} = (s, 0, c)$ on the positive x-axis. (See FIGURE 5.)

Let $\mathbf{P} = (X, Y, Z)$ represent an arbitrary point on the reflector. We need homogeneous coordinates for the tangent line to the curve at **P**. Therefore, let $\mathbf{P}_1$ be a second point on the curve, and consider the secant joining **P** and $\mathbf{P}_1$ whose homogeneous coordinates, according to formula (3), are

$$\mathbf{P}_1 \times \mathbf{P} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ X_1 & Y_1 & Z_1 \\ X & Y & Z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ X_1 - X & Y_1 - Y & z_1 - Z \\ X & Y & Z \end{vmatrix}.$$

Taking the limit as $\mathbf{P}_1$ approaches **P**, we get $\mathbf{T} = \mathbf{P}' \times \mathbf{P}$ where $\mathbf{P}' = (X', Y', Z')$.

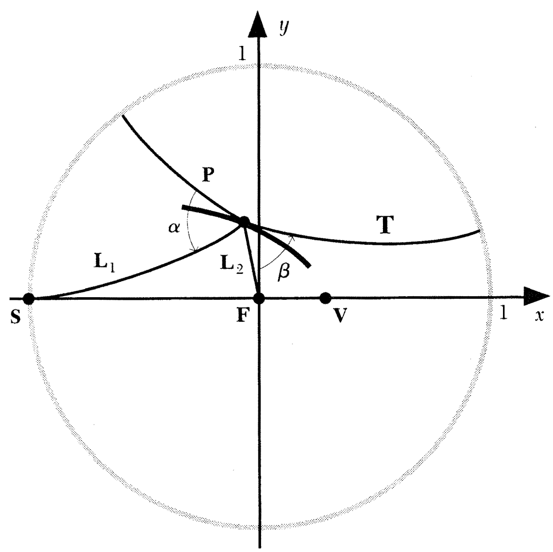

FIGURE 6
Points **P** such that $\alpha = \beta$ form a parabolic reflector.

The defining condition of the reflector is that the angles α and β in FIGURE 6 are equal. Formula (7) can be used to find these angles. For this purpose, let $\mathbf{L}_1$ be the ray from **S** to **P**, and let $\mathbf{L}_2$ be the line from **F** to **P**. Then

$$\cos(\alpha) = \left| \frac{\langle \mathbf{L}_1, \mathbf{T} \rangle}{\|\mathbf{L}_1\| \|\mathbf{T}\|} \right|, \quad \text{and} \quad \cos(\beta) = \left| \frac{\langle \mathbf{L}_2, \mathbf{T} \rangle}{\|\mathbf{L}_2\| \|\mathbf{T}\|} \right|$$

where $\mathbf{L}_1 = \mathbf{S} \times \mathbf{P} = [-Y, X + Z, -Y]$, and $\mathbf{L}_2 = \mathbf{F} \times \mathbf{P} = [-Y, X, 0]$. Using the easily verified formula

$$\langle \mathbf{A} \times \mathbf{B}, \mathbf{C} \times \mathbf{D} \rangle = \begin{vmatrix} \langle \mathbf{A}, \mathbf{C} \rangle & \langle \mathbf{B}, \mathbf{D} \rangle \\ \langle \mathbf{A}, \mathbf{D} \rangle & \langle \mathbf{B}, \mathbf{C} \rangle \end{vmatrix},$$

we get

$$\langle \mathbf{L}_1, \mathbf{T} \rangle = \langle \mathbf{S} \times \mathbf{P}, \mathbf{P}' \times \mathbf{P} \rangle = \langle \mathbf{S}, \mathbf{P}' \rangle \langle \mathbf{P}, \mathbf{P} \rangle - \langle \mathbf{S}, \mathbf{P} \rangle \langle \mathbf{P}, \mathbf{P}' \rangle$$

and

$$\langle \mathbf{L}_2, \mathbf{T} \rangle = \langle \mathbf{F} \times \mathbf{P}, \mathbf{P}' \times \mathbf{P} \rangle = \langle \mathbf{F}, \mathbf{P}' \rangle \langle \mathbf{P}, \mathbf{P} \rangle - \langle \mathbf{F}, \mathbf{P} \rangle \langle \mathbf{P}, \mathbf{P}' \rangle.$$

We now assume that $\mathbf{P}$ is normalized, so that $\langle \mathbf{P}, \mathbf{P} \rangle = 1$. By differentiation, it follows additionally that $\langle \mathbf{P}, \mathbf{P}' \rangle = 0$, so now $\langle \mathbf{L}_1, \mathbf{T} \rangle = \langle \mathbf{S}, \mathbf{P}' \rangle = Z' + X'$ and $\langle \mathbf{L}_2, \mathbf{T} \rangle = \langle \mathbf{F}, \mathbf{P}' \rangle = Z'$. Thus,

$$\cos(\alpha) = \left| \frac{Z' + X'}{Z + X} \right| \frac{1}{\|\mathbf{T}\|}$$

and

$$\cos(\beta) = \left| \frac{Z'}{\sqrt{X^2 + Y^2}} \right| \frac{1}{\|\mathbf{T}\|} = \left| \frac{Z'}{\sqrt{Z^2 - 1}} \right| \frac{1}{\|\mathbf{T}\|}.$$

Setting these equal, we get the differential equation

$$\pm \frac{Z' + X'}{Z + X} = \frac{Z'}{\sqrt{Z^2 - 1}}$$

which integrates to

$$\pm \ln|X + Z| = \ln\left|Z + \sqrt{Z^2 - 1}\right| + C.$$

The plus sign leads to a degenerate solution, but the negative sign, combined with the normalization condition ($Z^2 - X^2 - Y^2 = 1$) and initial conditions ($X_0 = s, Y_0 = 0$, $Z_0 = c$), yields the homogeneous equation

$$Y^2 = \frac{4}{(s + c)^2}(X + Z)(cZ - sX)$$

whose graph, in the disk model of the hyperbolic plane, is depicted in FIGURE 7.

What is the real hyperbolic parabola? There is no definitive answer to this question. Certainly both the curves derived above have important geometric properties characteristic of Euclidean parabolas. Both curves also have quadratic equations. Which do you prefer?

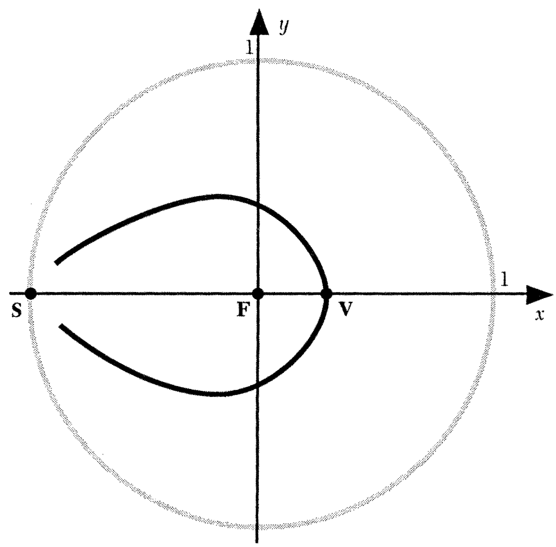

FIGURE 7
A parabolic reflector in the hyperbolic plane.

My own inclination is to regard the curve in Figure 7 as more genuinely parabolic than the one in Figure 4 because it tends, in the limit, to a *single* ideal point while the curve in FIGURE 4, like a Euclidean hyperbola, tends to *two* ideal points.

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1. H. S. M. Coxeter, *Non-Euclidean Geometry*, 5th Edition (1965), Univ. of Toronto Press.
2. Ron Perline, Non-Euclidean flashlights, *Amer. Math. Monthly*, 103 (1996), 377–385.

Math Bite:
The Volume of a Cone, Without Calculus or Square Roots

The volume V of a cone is a third the base A times the height H. M. Hirschhorn **[1]** obtains this relationship, and from it the volume and surface of a sphere, without calculus but with many square roots. Here we eliminate the square roots, and moreover obtain $3V = AH$ for any cone whose base is dissectable into squares whose total area, summed as a convergent series, is that of the base. (For example, a circular base of unit area admits such a dissection, but not when riddled with a series of holes of area $1/4, 1/8, 1/16, \ldots$ centered on an enumeration of its rational points.) Such a dissection induces a dissection of the cone into pyramids, whence to prove $3V = AH$ for such a cone it suffices to do so for a pyramid.

Any pyramid of nonzero volume can be transformed into any other by an affinity, a linear transformation composed with a translation. Affinities preserve the ratio of volume to base-times-height, whence it suffices to demonstrate $3V = AH$ for a single pyramid. The pyramid formed by the center and one face of the unit cube does the job, having $V = 1/6$ by symmetry, $H = 1/2$, and $A = 1$.

The surface area of a sphere Hirschhorn uses the volume of the cone to derive the relationship $3V = RS$ between the volume V and surface area S of a sphere of radius R (easy), and also to derive the formula for V itself in terms of R (harder), thereby obtaining S.

An alternative to the harder step is to obtain S first and apply $3V = RS$ to get V. For S, observe that a cylinder of radius R containing the sphere, and truncated at each end where the sphere ends (the same cylinder Hirschhorn uses to obtain V), has area the perimeter $2\pi R$ times the length $2R$, namely $4\pi R^2$. We show that the sphere has the same area.

Orienting the axis of the cylinder vertically, pair up points on the two surfaces via perpendiculars to the axis. This pairs up very small rectangles on the cylinder with very small rectangles on the sphere. Any such rectangle P on the cylinder is wider than its counterpart Q on the sphere in the ratio R/r where r is the distance of Q from the axis. But Q is taller than P by the same ratio because the tilt of Q from the vertical equals the tilt of the radius OQ (O the center of the sphere) from the horizontal. So the paired rectangles have the same area, whence so do the whole surfaces.

REFERENCE

1. M. Hirschhorn, The volume of a cone, without calculus, this MAGAZINE 70 (1997), 295–296.

—VAUGHAN PRATT
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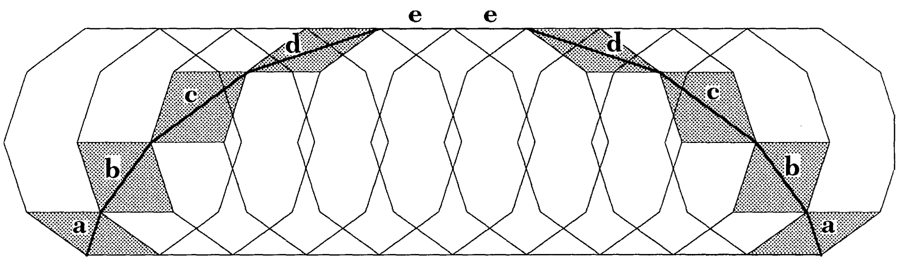
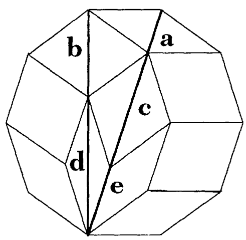
Proof Without Words: The Length of a Polygonal Arch

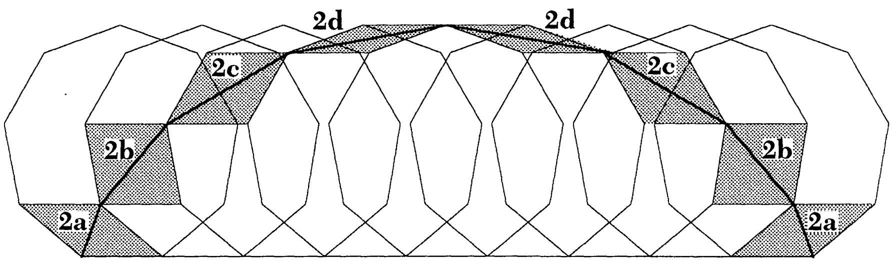
The length of the polygonal arch generated by one vertex of a regular n-gon rolling along a straight line is four times the length of the inradius plus four times the length of the circumradius of the n-gon.

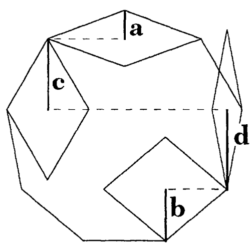
If n is even...





If n is odd...





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The Incredibly Knotty Checkerboard Challenge

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Introduction This note is about an easily-stated puzzle that arose in an interesting way. The puzzle involves certain patterns of coins on an ordinary checkerboard, but it was suggested by a theorem in diagrammatic knot theory! We begin by presenting the checkerboard puzzle. After we analyze the puzzle mathematically, we discuss the somewhat surprising connection between patterns of coins on a checkerboard and *knots*, which are simple, closed curves in space.

A standard checkerboard consists of sixty-four squares, arranged in eight rows and eight columns. If we focus on the corners of squares, then a checkerboard provides eighty-one vertices, arranged in a 9×9 grid.

By placing coins at some of these vertices, form any pattern that satisfies these conditions:

1. No coin lies at a vertex on the main diagonal of the board. (The main diagonal runs from upper-left to lower-right.)
2. The pattern is symmetric with respect to reflection across the main diagonal.

There are thus 2^{36} *admissible patterns*; see FIGURE 1 for an example.

Here is the challenge: By completely covering eight or fewer columns of vertices on the checkerboard, leave an even number—zero is allowed—of coins visible in each row.

For example, if we cover columns 2 and 8 in FIGURE 1, we obtain the pattern shown in FIGURE 2. Because an odd number of coins remain visible in each of four rows, this

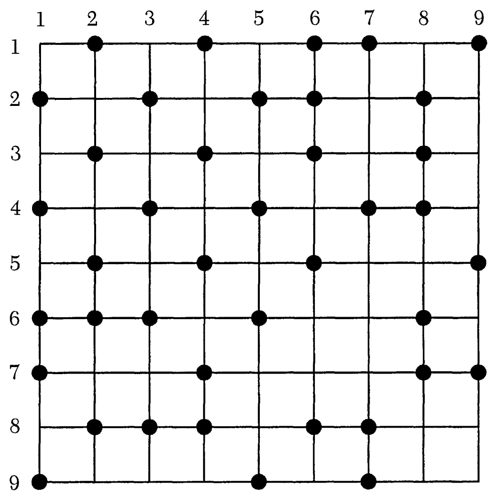

FIGURE 1
An admissible pattern.

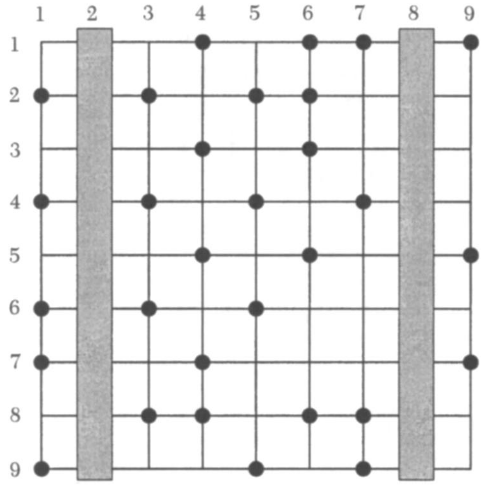

FIGURE 2
Covering columns.

is *not* a solution to the challenge. To solve the puzzle, we must cover columns to leave an even number of coins visible in *each* row.

In fact, there is a unique solution to the challenge for the pattern shown in FIGURE 1. We urge the reader to seek this solution! As a few minutes spent searching will confirm, finding the solution to this puzzle by "trial and error" is difficult, so there is something to be gained by confronting the challenge systematically. The primary aim of this note is to provide a systematic method for tackling these checkerboard puzzles.

Although we have begun by presenting a specific example, we shall show that not only does a solution exist for the pattern depicted in FIGURE 1, but that a solution exists for each admissible pattern of coins. Specifically, we shall prove:

THEOREM (Existence of Solutions). *Let n be an odd number. On an $n \times n$ grid, consider any pattern of coins that satisfies conditions* (1) *and* (2) *above. Then it is possible to cover j columns, where $0 \leq j \leq n-1$, to leave an even number of coins visible in each row.*

Notice we are not claiming each admissible pattern admits just one solution. Indeed, it is not difficult to find patterns of coins that admit multiple solutions. Also, according to the theorem, we don't need to confine ourselves to a standard checkerboard when we present this challenge. All we require is a square grid composed of an odd number of vertices.

In addition to proving the existence theorem stated above, we shall provide an algorithm for finding the solution or solutions whose existence the theorem guarantees. Actually, once the puzzle has been reformulated properly, it probably will be obvious to all which algorithm solves it. So a main goal of this note is to reformulate the checkerboard challenge to make it more amenable to mathematical investigation. Once this has been done, it will be rather easy to prove the existence theorem stated above, and to solve any given puzzle using a simple and well-understood algorithm.

Reformulating the challenge As perhaps the reader has discovered, when columns 2, 6, and 7 are covered in FIGURE 1, an even number of coins remain visible in each row. Let us use this solution to reformulate the challenge and make the puzzle much easier to analyze. The first step is an obvious one—use a 9×9 matrix to represent the pattern of coins on the checkerboard. With 1 representing a coin and 0 denoting an empty vertex, the pattern in FIGURE 1 corresponds to the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

With A_j denoting the jth column vector of A, we see that the solution given above *is* a solution precisely because

$$A_1 + A_3 + A_4 + A_5 + A_8 + A_9 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}.$$

or, equivalently, because

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}.$$

Of course, all that matters here is that the vector on the right-hand side of the equation contains even numbers only. Because of this, it is natural to abandon the real number system, and use instead the field $\mathbb{Z}/2\mathbb{Z}$. This field has only two elements, 0 and 1, which correspond to even and odd. In $\mathbb{Z}/2\mathbb{Z}$, the matrix equation above becomes

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now it should be clear that the checkerboard challenge corresponds to a standard problem in linear algebra! Indeed, finding the set of solutions for a given admissible pattern is equivalent to finding the nullspace of the matrix that represents the pattern. The only twist is that we must view the matrix as a matrix over $\mathbb{Z}/2\mathbb{Z}$, and we must view the solution vectors as elements of $(\mathbb{Z}/2\mathbb{Z})^9$ rather than $\mathbb{R}^9$. In theory, this change of fields matters little—we can find the solutions that correspond to a given pattern of coins by converting the matrix to row-echelon form and solving the associated homogeneous system of linear equations. In practice, using the two-element field makes computation quite easy, since row reduction of matrices over $\mathbb{Z}/2\mathbb{Z}$ is especially simple.

To illustrate, consider the matrix A that corresponds to the pattern of coins from FIGURE 1. Row reduction of that matrix over $\mathbb{Z}/2\mathbb{Z}$ yields the matrix

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since only the ninth column lacks a pivot, the nullspace of B is one-dimensional. So besides the zero vector, which doesn't correspond to a valid solution to the puzzle, there is a single vector X in the nullspace of B. By solving the associated homogeneous system of linear equations, we find the components of X are $x_2 = x_6 = x_7 = 0$ and $x_1 = x_3 = x_4 = x_5 = x_8 = x_9 = 1$. So we have recovered algebraically what we already knew—to solve the puzzle for the pattern of coins in FIGURE 1, we must cover columns 2, 6, and 7. Also, we have justified our assertion that this is the only solution to the puzzle for that pattern of coins.

Existence of solutions In this section, we prove the existence theorem stated in the Introduction. Actually, we prove the following reformulation of that theorem:

THEOREM (Existence of Solutions). *Let n be an odd number. Over the field $\mathbb{Z}/2\mathbb{Z}$, consider any $n \times n$ matrix that has the following properties:*

1. *Each element on the main diagonal of the matrix is* 0.
2. *The matrix is symmetric.*

Then there exists a non-zero vector in the nullspace of the matrix.

(A note on terminology: Henceforth, we will call an $n \times n$ matrix over $\mathbb{Z}/2\mathbb{Z}$ that satisfies 1) and 2) an *alternating matrix*, regardless of the parity of n. This usage is standard, and is appropriate since each such matrix represents an alternating bilinear form on $(\mathbb{Z}/2\mathbb{Z})^n$. We will call an odd-dimensional alternating matrix a *checkerboard matrix*. With this terminology, our theorem becomes "a checkerboard matrix has a non-trivial nullspace.")

Before we prove the theorem, two comments: First, n *must* be an odd number. Otherwise we can easily find alternating matrices that have trivial nullspaces. Indeed, for each even number n, the matrix that has ones immediately above and below the main diagonal and zeros elsewhere is such a matrix. Second, the theorem says nothing about uniqueness of solutions. In general, solutions are not unique, but they do form a subspace of $(\mathbb{Z}/2\mathbb{Z})^n$.

Now let us prove the theorem. With our reformulation, the proof is rather easy.

Proof. Let A be an $n \times n$ checkerboard matrix. It suffices to show that $\det(A) = 0$ in $\mathbb{Z}/2\mathbb{Z}$. Recall that $\det(A)$ is a certain alternating sum of products of elements of A (or see p. 514 in **[3]**). In $\mathbb{Z}/2\mathbb{Z}$, $-1 = +1$, so this alternating sum becomes an ordinary sum. Call a configuration of n locations in A a *transversal* if each row and each column of A is represented just once by the locations that comprise the configuration. There are thus $n!$ transversals, and each gives rise to a summand in $\det(A)$. Specifically, each transversal contributes to $\det(A)$ the product of the elements that occupy the locations that comprise the transversal. We must demonstrate that the sum of these $n!$ contributions is zero in $\mathbb{Z}/2\mathbb{Z}$.

To see this sum is zero, note that any transversal that contains at least one location on the main diagonal contributes nothing, since each element on the diagonal of A is zero. Those transversals that do not intersect the main diagonal occur in pairs—each such transversal has a "mirror image" obtained by reflection across the main diagonal. (Because n is odd, no such transversal is its own mirror image.) Since A is symmetric, each such transversal and its mirror image make the same contribution; since we are working in $\mathbb{Z}/2\mathbb{Z}$, the combined contribution of the paired transversals is zero. Thus $\det(A) = 0$.

A connection to knot theory In this section, we describe the path we followed from knot theory to the checkerboard challenge. The account here is self-contained, but for more about knot theory, see **[1]**, **[2]**, or **[4]**.

As above, we use an example to focus our discussion. A diagram for the knot 9_{16} is shown in FIGURE 3. This is a *positive knot diagram*, which means each crossing in the diagram is a positive crossing according to the conventions of knot theory. (See p. 152 of **[1]** for a discussion of crossing signs.) We have assigned a number to each crossing; this has been done in an arbitrary manner, and simply for reference.

To this labeled knot diagram, we assign a 9×9 matrix A over $\mathbb{Z}/2\mathbb{Z}$, as follows: Let $a_{ii} = 0$ for $i = 1, 2, \ldots, 9$ and, for $i \neq j$, let a_{ij} be the number of times (mod 2) that we traverse crossing i when we trace along the knot from overcrossing j to

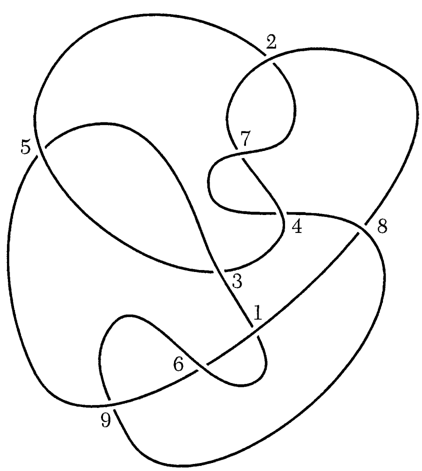

FIGURE 3
The knot 9_{16}.

undercrossing j. For example, to determine the first column of A, we start at the overcrossing labeled 1 and trace along the knot diagram (in either direction) until we reach the undercrossing labeled 1. During this trip, we traverse crossings 3, 5, 6, and 9 once, and we traverse crossings 2, 4, 7, and 8 either twice or not at all. Thus $a_{21} = a_{41} = a_{71} = a_{81} = 0$ and $a_{31} = a_{51} = a_{61} = a_{91} = 1$. By following the same procedure for each of the remaining eight columns, we obtain the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is called the *trip matrix* of the knot diagram, and it contains much information about the knot that the diagram represents. In particular, we can obtain the Jones polynomial of the knot directly from the matrix. (The Jones polynomial is a knot invariant that often can distinguish one knot from another.) For more about trip matrices, see **[5]**. Notice our trip matrix is an alternating matrix—in particular, it is symmetric. However, as we shall see, not every alternating matrix is the trip matrix of a positive knot diagram.

In **[6]**, it is proved that the rank over $\mathbb{Z}/2\mathbb{Z}$ of the trip matrix of a positive knot diagram is an *even* number. Specifically, the rank is shown to be twice the genus of the simplest orientable surface that spans the knot. (See Section 4.3 of **[1]** for more about spanning surfaces of knots.) Thus, for the trip matrix of a positive knot diagram with an *odd* number of crossings, the nullspace must be non-trivial. This is an immediate consequence of "rank plus nullity equals size of matrix." So, for patterns of coins that correspond to such knots, we knew *a priori* that solutions had to exist. The next question was natural: What about checkerboard matrices that don't correspond to knots—must they also have non-trivial nullspaces? With that question, this note was born.

As we know now, the answer to that question is yes. In fact, the answer is also yes to another natural question: Does each alternating matrix have even rank over $\mathbb{Z}/2\mathbb{Z}$? As noted above, for trip matrices of positive knot diagrams this can be proved using knot-theoretic arguments. To prove the result in general, we can view our matrix as the matrix (with respect to the standard basis) of an alternating bilinear form on $(\mathbb{Z}/2\mathbb{Z})^n$, and invoke Theorem 8.1 on p. 586 of **[3]**. This theorem provides a canonical decomposition of each alternating bilinear form, and an examination of this decomposition immediately shows that the rank of each matrix that represents the form is even. Of course, the affirmative answer to the second natural question implies an affirmative answer to the first, giving another—albeit indirect—proof of our theorem.

In retrospect, what is amazing about the birth of the checkerboard challenge is what we *didn't* see. Our result from knot theory told us that the trip matrix of a positive knot diagram with an odd number of crossings had a non-trivial nullspace, or equivalently, that the corresponding checkerboard puzzle had a solution. But it wasn't until much later that we noticed the obvious: Meeting the checkerboard challenge for a pattern of coins that corresponds to *any* positive knot diagram is easy. In fact, we don't have to cover columns at all, the pattern comes already solved! This is because, if we trace along a knot diagram from an overcrossing to the corresponding undercrossing, we will traverse an even number of other crossings, counting multiplicities. This is true no matter the number of crossings in the diagram, so for patterns that correspond to positive knots, parity is not an issue at all. This also shows there are an infinite number of alternating matrices that do not correspond to positive knot diagrams.

There is something else that is interesting about patterns that correspond to positive knot diagrams. Not only do such patterns come already solved, but we can use the knot diagram to find any additional solutions that might exist. We do this as follows: We begin by placing an orientation on the knot. Then we simply replace each crossing in the diagram with a pair of small, uncrossed arcs that preserve the orientation. See FIGURE 4, which shows the result of this process for the knot from FIGURE 3.

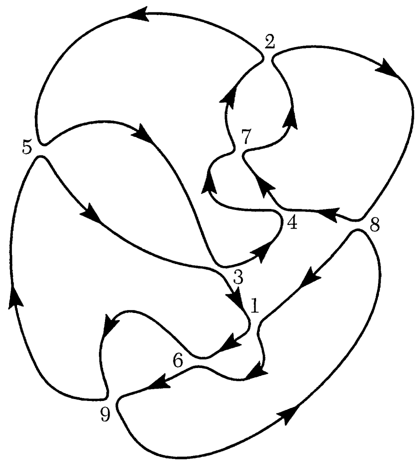

FIGURE 4
Seifert circles.

The topological circles that result are called the *Seifert circles* of the oriented knot diagram, and they encode a collection of solutions to our puzzle. Each Seifert circle approaches sites formerly occupied by crossings. Each of those sites retains a numerical label, which identified the now-departed crossing. Thus, from each Seifert circle, we can obtain a set of numbers. For example, from the Seifert circles shown in FIGURE 4, we obtain $\{1,3,5,6,9\}$, $\{2,3,4,5,7\}$, $\{2,4,7,8\}$ and $\{1,6,8,9\}$. Each of these sets yields a solution to the puzzle for the pattern of coins that corresponds to the knot 9_{16}. For example, $\{1,3,5,6,9\}$ indicates that we can obtain a solution by leaving those five columns *uncovered* on the board. The other three sets yield additional solutions in the same way, so here the Seifert circles are directly providing four solutions. Since the sum of these solutions corresponds to the zero vector in $(\mathbb{Z}/2\mathbb{Z})^9$, the solutions are not linearly independent over $\mathbb{Z}/2\mathbb{Z}$. However, any three of them are independent and form a basis for the space of solutions. Thus, in total, there are seven solutions to the puzzle for this knot diagram. These seven solutions correspond to the non-trivial linear combinations over $\mathbb{Z}/2\mathbb{Z}$ of any three basic solutions.

In fact, what we have demonstrated for the knot 9_{16} is true in general. To find a basis for the set of solutions to the puzzle that corresponds to a positive knot diagram, simply resolve the diagram into Seifert circles and discard any circle. The sets that correspond to the circles that remain provide a basis for the set of solutions to the puzzle. (This can be seen by examining the proof of Theorem 2 in **[5]**.) So, at least for certain special patterns of coins, we don't even need linear algebra to meet the incredible checkerboard challenge. We simply must draw the right picture!

Acknowledgment. We thank the anonymous referees for several valuable suggestions. We also thank Davide Cervone, for producing the figures and for a discussion that made this a better note.

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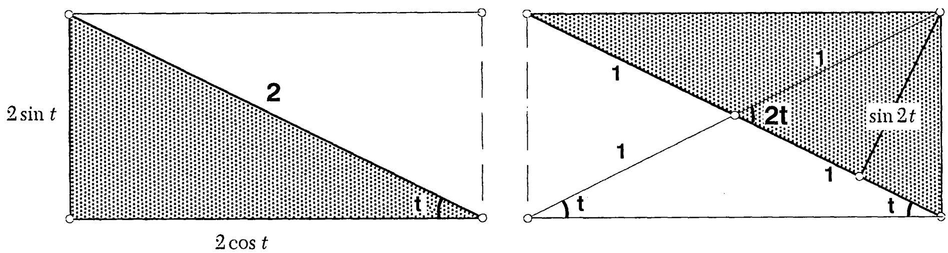
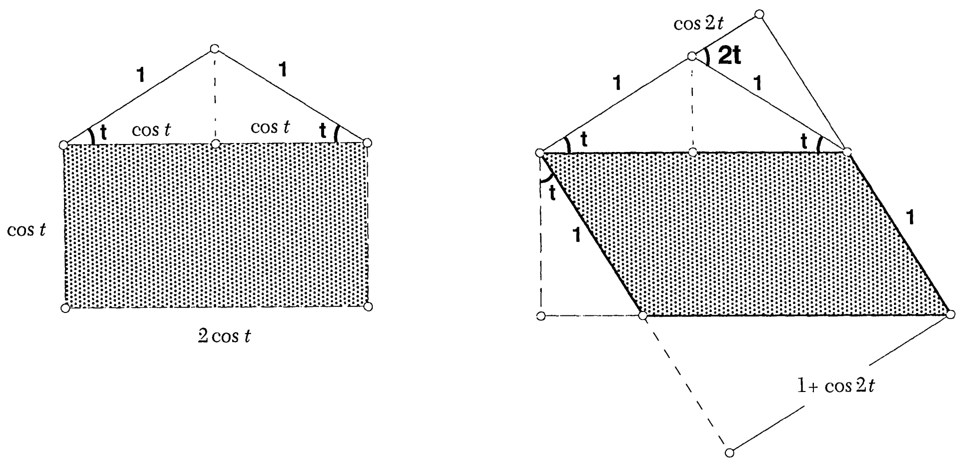
Proof Without Words: Double Angle Formulas



$2 \sin t \cos t \quad = \quad \sin 2t$



$2 \cos^2 t \quad = \quad 1 + \cos 2t$

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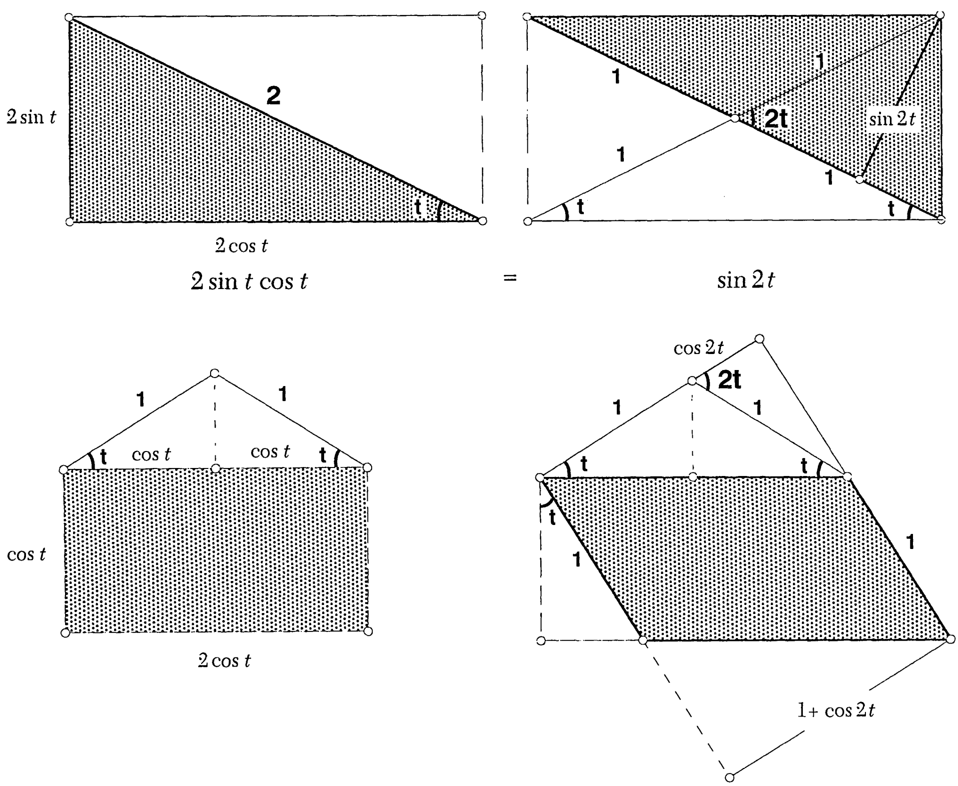

$2\sin t\cos t = \sin 2t$

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Two Diophantine Equations Studied by Ramanujan

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Introduction In his second notebook [**3**, p. 225], Ramanujan wrote

$$\left(6n^2 + (3n^3 - n)\right)^3 + \left(6n^2 - (3n^3 - n)\right)^3 = \left(6n^2(3n^2 + 1)\right)^2 \qquad (1)$$

and

$$\left(m^7 - 3m^4(1+p) + m\left(3(1+p)^2 - 1\right)\right)^3 + \left(2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\right)^3$$
$$+\left(m^6 - (1+3p+3p^2)\right)^3$$
$$= \left(m^7 - 3m^4p + m(3p^2 - 1)\right)^3. \qquad (2)$$

These represent solutions of the diophantine equations

$$A^3 + B^3 = C^2 \qquad (3)$$

and

$$A^3 + B^3 + C^3 = D^3. \qquad (4)$$

It appears ([**2**, pp. 578–9]) that equation (3) was first studied by Euler, and the general solution found by R. Hoppe. According to ([**2**, pp. 550–554]), equation (4) was studied by Vieta and Fermat, and the general solution found by Euler. Both equations, especially (4), have aroused considerable interest over the centuries. But Ramanujan knew none of this when he composed his notebooks [**3**].

It is, of course, easy to verify Ramanujan's solutions. But the question arises: How did Ramanujan obtain them? My object is to give a plausible answer.

The diophantine equation $A^3 + B^3 = C^2$ Suppose that $A^3 + B^3 = C^2$, and write $A = x + y$ and $B = x - y$. Then

$$A^3 + B^3 = (x+y)^3 + (x-y)^3 = 2x^3 + 6xy^2 = 2x(x^2 + 3y^2).$$

Now write $y = a - b$ and $x^2 = 12ab$, for then $x^2 + 3y^2 = 3(a+b)^2$ and $A^3 + B^3 = 6x(a+b)^2$. Now let $x = 6m^2$. Then

$$A^3 + B^3 = 36m^2(a+b)^2 = (6(a+b)m)^2 = C^2,$$

with $C = 6(a+b)m$. Also, $12ab = x^2 = (6m^2)^2 = 36m^4$, so $ab = 3m^4$. Thus we obtain the following result.

THEOREM 1. *If m is an integer and if $ab = 3m^4$, then*

$$\left(6m^2 + (a-b)\right)^3 + \left(6m^2 - (a-b)\right)^3 = \left(6(a+b)m\right)^2.$$

Indeed, a straightforward calculation yields the identity

$$(6m^2+(a-b))^3+(6m^2-(a-b))^3=(6(a+b)m)^2+144m^2(3m^4-ab)$$

of which Theorem 1 is an immediate corollary.

If we set $a=3m^2n$, $b=m^2/n$, multiply by n^6 and delete the factor m^6, we obtain Ramanujan's result (1).

If, instead, we set $a=3m^4/n^2$ and $b=n^2$, and multiply by n^6, we obtain Euler's solution of (3):

$$(3m^4+6m^2n^2-n^4)^3+(-3m^4+6m^2n^2+n^4)^3=(18m^5n+6mn^5)^2.$$

Yet again, if we set $a=3n$ and $b=m^4/n$, and multiply by n^6, we obtain

$$(3n^3+6m^2n^2-m^4n)^3+(-3n^3+6m^2n^2+m^4n)^3=(18mn^4+6m^5n^2)^2.$$

If we now set $n=m^2-c$, we obtain

$$(8m^6-20cm^4+15c^2m^2-3c^3)^3+(4m^6-4cm^4-3c^2m^2+3c^3)^3$$
$$=(24m^9-84cm^7+114c^2m^5-72c^3m^3+18c^4m)^2.$$

Now put $c=2p$ and divide throughout by $4^3=8^2$, to find

$$(m^6-2pm^4-3p^2m^2+6p^3)^3 4+(2m^6-10pm^4+15p^2m^2-6p^3)^3$$
$$=(3m^9-21pm^7+57p^2m^5-72p^3m^3+36p^4m)^2.$$

The diophantine equation $A^3+B^3+C^3=D^3$ Suppose that $A^3+B^3+C^3=D^3$, and write $A=y-x$, $B=u+v$, $C=u-v$, and $D=y+x$. Then $u(u^2+3v^2)=x(x^2+3y^2)$, or, equivalently,

$$\frac{u}{x}=\frac{x^2+3y^2}{u^2+3v^2}.$$

If we set both sides equal to M, we find $u=Mx$ and $x^2+3y^2=M(u^2+3v^2)=M(Mx)^2+3Mv^2=M^3x^2+3Mv^2$, or, equivalently,

$$3(y^2-Mv^2)=(M^3-1)x^2.$$

Now we let $M=m^2$, to obtain $3(y+mv)(y-mv)=(m^6-1)x^2$.

Up to this point I have been heavily indebted to Bruce Berndt [**1**, p. 198], but now our paths diverge. We set $x=3m$, to get $(y+mv)(y-mv)=3m^2(m^6-1)$. Now we write $y+mv=am$ and $y-mv=bm$, where $ab=3(m^6-1)$. Then

$$y=\frac{1}{2}(am+bm) \quad \text{and} \quad v=\frac{1}{2}(a-b),$$

with $x=3m$ and $u=3m^3$. Thus we have

$$A=\frac{1}{2}(a+b-6)m, \quad B=\frac{1}{2}(6m^3+(a-b))$$
$$C=\frac{1}{2}(6m^3-(a-b)), \quad D=\frac{1}{2}(a+b+6)m,$$

and we obtain the following result:

THEOREM 2. *If m is an integer and if $ab = 3(m^6 - 1)$, then*

$$((a+b-6)m)^3 + (6m^3 + (a-b))^3 + (6m^3 - (a-b))^3 = ((a+b+6)m)^3.$$

Again, a straightforward calculation yields

$$((a+b-6)m)^3 + (6m^3 + (a-b))^3 + (6m^3 - (a-b))^3$$
$$= ((a+b+6)m)^3 + 144m^3(3(m^6-1) - ab),$$

of which Theorem 2 is an immediate corollary.

If we write $a = 3n$ and $b = (m^6 - 1)/n$, and multiply through by n^3 we obtain

$$(3mn^2 - 6mn + m^7 - m)^3 + (3n^2 + 6m^3n - m^6 + 1)^3 + (-3n^2 + 6m^3n + m^6 - 1)^3$$
$$= (3mn^2 + 6mn + m^7 - m)^3.$$

Next, we set $n = m^3 - c$ and obtain

$$(4m^7 - (6c+6)m^4 + (3c^2 + 6c - 1)m)^3$$
$$+ (8m^6 - 12cm^3 + (3c^2+1))^3 + (4m^6 - (3c^2+1))^3$$
$$= (4m^7 - (6c-6)m^4 + (3c^2 - 6c - 1)m)^3.$$

Finally, we put $c = 2p + 1$ and divide throughout by 4^3, to find

$$(m^7 - (3p+3)m^4 + (3p^2 + 6p + 2)m)^3$$
$$+ (2m^6 - (6p+3)m^3 + (3p^2 + 3p + 1))^3$$
$$+ (m^6 - (3p^2 + 3p + 1))^3$$
$$= (m^7 - 3pm^4 + (3p^2 - 1)m)^3,$$

which is Ramanujan's result (2).

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PROBLEMS

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Proposals

To be considered for publication, solutions should be received by May 1, 1999.

1559. *Proposed by Joaquín Gómez Rey, I. B. "Luis Buñuel," Alcorcón, Madrid, Spain.*

For what complex numbers z is the sequence $(a_n(z))_{n\geq 0}$ defined by

$$a_n(z) = \sum_{k=0}^{n} \binom{n+k}{2k} z^k$$

periodic?

1560. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, Guang Dong Province, China.*

Points A, B, C, P, Q, and T lie on a circle and satisfy $AB > AC$, T is on the same side of BC as A with $TB = TC$, and $AP = AQ = \sqrt{AB \cdot AC}$. Let $[ABC]$ denote the area of $\triangle ABC$, and so forth.

(a) If $\angle BAC \geq 90°$, prove that $[ABC] > [APQ]$.

(b) If $\sqrt{AB \cdot AC} \leq BC$, prove that $[TBC] > [APQ]$.

1561. *Proposed by Emre Alkan, student, University of Wisconsin, Madison, Wisconsin.*

Let $a_1, \ldots, a_k$ be pairwise relatively prime, positive integers. Determine the largest integer not expressible in the form

$$x_1 a_2 a_3 \cdots a_k + x_2 a_1 a_3 \cdots a_k + x_k a_1 a_2 \cdots a_{k-1},$$

for some nonnegative integers $x_1, \ldots, x_k$.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. *Each solution should begin on a separate sheet containing the solver's name and full address.*

Solutions and new proposals should be mailed to George T. Gilbert, Problems Editor, Department of Mathematics, Box 298900, Texas Christian University, Fort Worth, TX 76129, or mailed electronically (ideally as a LATEX file) to g.gilbert@tcu.edu. *Readers who use e-mail should also provide an e-mail address.*

1562. *Proposed by John Wickner, student, St. Thomas College, St. Paul, Minnesota, and Scott Beslin and Valerio De Angelis, Nicholls State University, Thibodaux, Louisiana.*

Prove that

$$\tan\left(\frac{1}{4}\tan^{-1}4\right)2\left(\cos\frac{6\pi}{17}+\cos\frac{10\pi}{17}\right).$$

1563. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, Guang Dong Province, China.*

For a given field F, classify all possible partitions of F into finitely many equivalence classes such that each class is closed under addition and multiplication by *distinct* elements in the class.

Quickies

Answers to the Quickies are on pages 396.

Q885. *Proposed by Keivan Mallahi, student, Sharif University of Technology, Tehran, Iran.*

Let S be a finite set of $n\times n$ matrices over a field. If S is closed under multiplication, prove that there exists $M\in S$ such that $\text{Trace}(M)\in\{0,1,2,\ldots,n\}$.

Q886. *Proposed by Zheng-Ping Tian, Hangzhou Teacher's College, Hangzhou, Zhejiang, China.*

If a, b, and c are real numbers that satisfy $a\geq b\geq c\geq 0$ and $a+b+c=3$, show that $ab^2+bc^2+ca^2\leq 27/8$.

Solutions

Sum of a Sequence of Floors and Ceilings — December 1997

1534. *Proposed by Donald Knuth, Stanford University, Stanford, California.*

Let m, n, and p be positive integers, and set

$$t_{m,p}(n)=\left\lceil\frac{\lfloor n/m\rfloor}{2p}\right\rceil,\quad s_{m,p}(n)=t_{m,p}(0)+t_{m,p}(1)+\cdots+t_{m,p}(n-1).$$

Prove that $s_{m,p}(n)$ is a multiple of $t_{m,p}(n)$.

Solution by Matthias Beck and Akalu Tefera, Temple University, Philadelphia, Pennsylvania, and Melkamu Zeleke, The William Paterson University of New Jersey, Wayne, New Jersey.

We prove a slightly more general result: Let $m,n,p\in\mathbb{N}$ and define $T_{m,p}(n):=\lceil\lfloor n/m\rfloor/p\rceil$ and $S_{m,p}(n):=\sum_{j=0}^{n-1}T_{m,p}(j)$. Then $S_{m,p}(n)$ is a multiple of $T_{m,p}(n)$ if and only if $n \le m$ or at least one of the integers m, p, q is even, where $q := \lfloor (n-m)/(pm) \rfloor$.

To begin the proof, observe that if $n \le m$, then $S_{m,p}(n) = 0$, which is clearly a multiple of $T_{m,p}(n)$. Therefore, assume $n > m$. Then

$$T_{m,p}(n) = \left\lceil \frac{\lfloor n/m \rfloor}{p} \right\rceil = \left\lfloor \frac{\lfloor n/m \rfloor + p - 1}{p} \right\rfloor = \left\lfloor \frac{\lfloor n/m - 1 \rfloor}{p} \right\rfloor + 1$$
$$= \left\lfloor \frac{n-m}{pm} \right\rfloor + 1 = q + 1 > 0,$$

where we used the fact that, for $b \in \mathbb{N}$, $\lfloor \lfloor a \rfloor / b \rfloor = \lfloor a/b \rfloor$. Furthermore,

$$S_{m,p}(n) = \sum_{j=0}^{n-1} T_{m,p}(j) = \sum_{j=0}^{n-1} \left(\left\lfloor \frac{j-m}{pm} \right\rfloor + 1 \right).$$

Now divide $[m, n-1]$ into subintervals of pm integers (plus the remaining final subinterval, which could be empty), each representing a constant contribution to $S_{m,p}(n)$. Thus, we have

$$S_{m,p}(n) = \sum_{j=0}^{m-1} 0 + \sum_{k=0}^{q-1} \sum_{j=pmk+m}^{pm(k+1)+m-1} (k+1) + \sum_{j=pmq+m}^{n-1} (q+1)$$
$$= \sum_{k=0}^{q-1} pm(k+1) + (n - pmq - m)(q+1)$$
$$= \frac{pmq(q+1)}{2} + (n - pmq - m)(q+1) = \left(n - m - \frac{pmq}{2} \right) T_{m,p}(n).$$

It follows that $T_{m,p}(n)$ divides $S_{m,p}(n)$ if and only if $n - m - pmq/2$ is an integer, which in turn holds if and only if pmq is even.

Also solved by J. C. Binz (Switzerland), Mark Bowron, David Callan, John Christopher, Con Amore Problem Group (Denmark), Daniele Donini (Italy), Marty Getz and Dixon Jones, Thomas Jager, Sean McIlroy (Canada), Ioana Mihaila, Kenneth Rogers, Heinz-Jürgen Seiffert (Germany), Nicholas C. Singer, The TAMUK Problem Solvers, Western Maryland College Problems Group, and the proposer. There were three incorrect solutions.

A Class of Real-Valued Functions on Groups — December 1997

1535. *Proposed by Sergei Ovchinnikov, San Francisco State University, San Francisco, California.*

Let S be a nonempty set of real numbers. Prove that there exists a group G and a surjective function $f: G \to S$ satisfying

$$f(xy^{-1}) \ge \min\{f(x), f(y)\} \quad \text{for all } x, y \in G$$

if and only if $\sup S \in S$.

Solution by Yan-loi Wong, The National University of Singapore, Singapore, Republic of Singapore.

Suppose that such a group G and a surjection $f: G \to S$ exist. Let $s \in S$. As f is surjective, there exists $x \in G$ such that $f(x) = s$. Denote the identity element of G by e. Then, $f(e) = f(xx^{-1}) \ge \min\{f(x), f(x)\} = f(x) = s$. Hence, $\sup S = f(e) \in S$.

Conversely, suppose $\sup S \in S$. Let G be the free abelian group generated by S. Then $x \in G$ can be represented as $\sum_{s \in S} n_s s$, where only finitely many of the integers n_s are nonzero. Define a function $f: G \to S$ as follows. For the identity element 0 of G, define $f(0) := \sup S$. For $x \neq 0$, define $f(x) := \min\{s \mid n_s \neq 0\}$. Clearly, f is surjective. It is straightforward to check that $f(x-y) \geq \min\{f(x), f(y)\}$ for all $x, y \in G$.

Also solved by Matt Baker (graduate student), Daniele Donini (Italy), Marty Getz and Dixon Jones, Thomas Jager, Michael Josephy (Costa Rica), John Koker, J. H. Pathak, W. R. Smythe, and the proposer.

Determinants of Catalan Numbers **December 1997**

1536. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, New York.*

Let $c_n = \binom{2n}{n}/(n+1)$ be the Catalan numbers. Evaluate the determinants

$$A_n = \begin{vmatrix} 1 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ -1 & 1 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ 0 & -1 & 1 & \cdots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & c_1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix} \quad \text{and}$$

$$B_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ -1 & 2 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ 0 & -1 & 2 & \cdots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 2 & c_1 \\ 0 & 0 & 0 & \cdots. & -1 & 2 \end{vmatrix}$$

I. *Solution by Frank A. Horrigan, Raytheon Systems Company, Tewksbury, Massachusetts.*

We show that $A_n = c_n$ and $B_n = c_{n+1}$.

Define $A_0 := 1$, $B_0 := 1$, and the generating functions $A(x) := \sum_{n=0}^{\infty} A_n x^n$, $B(x) := \sum_{n=0}^{\infty} B_n x^n$, and $C(x) := \sum_{n=0}^{\infty} c_n x^n$. First let us evaluate the generating function $C(x)$. A recursive relationship can be written from the definition of c_n, namely $(n+2)c_{n+1} = (4n+2)c_n$. Multiplying both sides of this recursion by x^n and summing from 0 to ∞, we find

$$\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} c_{n+1}x^n = 4\sum_{n=0}^{\infty} nc_n x^n + 2\sum_{n=0}^{\infty} c_n x^n,$$

or

$$\frac{d}{dx}C(x) + \frac{C(x)-1}{x} = 4x\frac{d}{dx}C(x) + 2C(x).$$

With initial condition $C(0) = c_0 = 1$, this differential equation has solution

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}, \quad C(0) = 1,$$

which satisfies

$$x[C(x)]^2 - C(x) + 1 = 0. \tag{1}$$

We now evaluate the determinants A_{n+1} and B_{n+1} through repeated expansion by minors using the left-most column at each stage, obtaining

$$A_{n+1} = A_n + \begin{vmatrix} c_1 & c_2 & \dots & c_{n-1} & c_n \\ -1 & 1 & \dots & c_{n-3} & c_{n-2} \\ & & \ddots & & \\ 0 & 0 & \dots & 1 & c_1 \\ 0 & 0 & \dots & -1 & 1 \end{vmatrix}$$

$$= \cdots = c_0 A_n + c_1 A_{n-1} + \cdots + c_{n-1} A_1 + c_n A_0$$

and

$$B_{n+1} = 2B_n + \begin{vmatrix} c_1 & c_2 & \dots & c_{\{}n-1 & c_n \\ -1 & 2 & \dots & c_{n-3} & c_{n-2} \\ & & \ddots & & \\ 0 & 0 & \dots & 2 & c_1 \\ 0 & 0 & \dots & -1 & 2 \end{vmatrix}$$

$$= \cdots = B_n + c_0 B_n + c_1 B_{n-1} + \cdots + c_{n-1} B_1 + c_n B_0.$$

Multiplying both sides of each equation by x^{n+1} and summing from 0 to ∞, we find that $A(x) - 1 = xA(x)C(x)$ and $B(x) - 1 = xB(x) + xB(x)C(x)$. Solving for $A(x)$ and $B(x)$ and using equation (1) yields

$$A(x) = \frac{1}{1 - xC(x)} = C(x) \quad \text{and} \quad B(x) = \frac{1}{1 - x - xC(x)} = \frac{C(x) - 1}{x}.$$

Therefore, $A_n = c_n$ and $B_n = c_{n+1}$.

II. *Solution by Lou Shapiro, Howard University, Washington, D. C.*

Equation (1) of Solution I above implies the standard identity

$$c_n = c_0 c_{n-1} + c_1 c_{n-2} + c_2 c_{n-3} + \cdots + c_{n-1} c_0. \tag{2}$$

We may rewrite this identity for n, $n-1, \dots, 1, 0$ in matrix form as

$$\begin{pmatrix} 1 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ -1 & 1 & c_1 & \dots & c_{n-3} & c_{n-2} \\ 0 & -1 & 1 & \dots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & c_1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1} \\ c_{n-2} \\ c_{n-3} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} c_n \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Applying Cramer's rule to solve for the final variable yields

$$1 = c_0 = \frac{\begin{vmatrix} 1 & c_1 & c_2 & \dots & c_{n-2} & c_n \\ -1 & 1 & c_1 & \dots & c_{n-3} & 0 \\ 0 & -1 & 1 & \dots & c_{n-4} & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ -1 & 1 & c_1 & \dots & c_{n-3} & c_{n-2} \\ 0 & -1 & 1 & \dots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & c_1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{vmatrix}} = \frac{c_n}{A_n}.$$

Because the determinant in the numerator, c_n, is non-zero, A_n is non-zero as well. This justifies the use of Cramer's rule and allows us to conclude that $A_n = c_n$.

Similarly, a slight manipulation of the identity (2) yields

$$\begin{pmatrix} 2 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ -1 & 2 & c_1 & \dots & c_{n-3} & c_{n-2} \\ 0 & -1 & 2 & \dots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 2 & c_1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \begin{pmatrix} c_n \\ c_{n-1} \\ c_{n-2} \\ \vdots \\ c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_{n+1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Cramer's rule now implies $1 = c_1 = c_{n+1}/B_n$, or $B_n = c_{n+1}$.

Also solved by Anchorage Math Solutions Group, J. C. Binz (Switzerland), Stan Byrd and Ronald L. Smith, David Callan, C. Coker, Con Amore Problem Group (Denmark), Daniele Donini (Italy), Seyoum Getu, Marty Getz and Dixon Jones, Thomas Jager, Harris Kwong, Carl Libis, Allan Pedersen (Denmark), Heinz-Jürgen Seiffert (Germany), William F. Trench, Western Maryland College Problems Group, Michael Woltermann, and the proposer. There was one incomplete solution.

A Nim-Type Game **December 1997**

1537. *Proposed by Jerrold W. Grossman, Oakland University, Rochester, Michigan.*

A two-person game is played as follows. A position consists of a pair (a, b) of positive integers. Players alternate moves, a move consisting of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. (This happens when $a = b$.) Determine those a and b such that the player who goes first from position (a, b) can guarantee a win with optimal play.

Solution by Philip D. Straffin, Beloit College, Beloit, Wisconsin.

The player who goes first can guarantee a win if and only if the ratio of the larger number to the smaller is greater than the "golden ratio" $\phi = (1 + \sqrt{5})/2$. To show this, we partition the set of unordered pairs of (not necessarily distinct) positive integers into the set $\mathscr{W}$ of pairs for which this condition is true, and the set $\mathscr{L}$ of pairs for which it is false. We must show that

(i) for any pair in $\mathscr{W}$, there is at least one move which leaves a pair in $\mathscr{L}$, and
(ii) for any pair in $\mathscr{L}$, all legal moves leave a pair in $\mathscr{W}$.

To prove (i), consider $\{a, b\}$ with $a/b > \phi$. It suffices to show that there is a positive integer k such that

$$\frac{1}{\phi} < \frac{a-bk}{b} < \phi.$$

Since ϕ satisfies the identity $1/\phi = \phi - 1$, this is equivalent to

$$\phi - 1 < \frac{a}{b} - k < \phi.$$

Because ϕ is irrational, there is exactly one such integer k, which is positive since $a/b > \phi$. We have shown that for any position in $\mathscr{W}$, there is exactly one move leaving a pair in $\mathscr{L}$.

To prove (ii), consider $\{a, b\}$ with $1 \le a/b < \phi$. If $a = b$, there is no legal move and the player to move has lost. If $a > b$, the only legal move is to $\{a-b, b\}$, and then

$$\frac{b}{a-b} = \frac{1}{\frac{a}{b}-1} > \frac{1}{\phi - 1} = \phi.$$

Also solved by Christian Blatter (Switzerland), David M. Bloom, Jean Bogaert (Belgium), David Callan, John Christopher, Con Amore Problem Group (Denmark), Daniele Donini (Italy), William Gasarch, Marty Getz and Dixon Jones, Robert Gibson, Peter Griffin, Thomas Jager, Kevin McDougal, Sean McIlroy (Canada), William A. Newcomb, Oklahoma State University Problem Solving Group, Allen J. Schwenk, Jorge-Nuno Silva (Portugal), W. R. Smythe, James A. Swenson (student), Michael Woltermann, and the proposer. There was one incorrect solution.

A 5th Degree, Symmetric Diophantine Equation **December 1997**

1538. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and George T. Gilbert, Texas Christian University, Fort Worth, Texas.*

Find all integer solutions to $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$.

I. *Solution by Brian D. Beasley, Presbyterian College, Clinton, South Carolina.*

We show that the given equation holds for integers x and y if and only if $x + y + 1 = 0$.

The given equation is true if and only if

$$2(x^5 + y^5 + 1) - 5xy(x^2 + y^2 + 1) = (x + y + 1)f(x, y) = 0,$$

where

$$f(x, y) = 2x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + 2y^4 - 2x^3 - x^2y - xy^2 - 2y^3 + 2x^2 - xy + 2y^2 - 2x - 2y + 2.$$

Thus we need only show that $f(x, y) \neq 0$ for all integers x and y. Observe that in any solution of the original equation, x and y must have opposite parity. By symmetry, we may assume without loss of generality that x is even and y is odd. Then

$$f(x, y) \equiv 2y^4 - xy^2 - 2y^3 - xy + 2y^2 - 2y + 2 \pmod 4.$$

However, each of the expressions $2y^4 - 2y^3 = 2y^3(y-1)$, $-xy^2 - xy = -xy(y+1)$, and $2y^2 - 2y = 2y(y-1)$ is divisible by 4 for x even, leaving $f(x, y) \equiv 2 \pmod 4$.

II. *Solution by Lenny Jones and students Karen Blount, Dennis Reigle, and Beth Stockslager, Shippensburg University, Shippensburg, Pennsylvania.*

The only solutions are ordered pairs of integers (x, y) with $x+y+1=0$.

To see this, factor $2(x^5+y^5+1)-5xy(x^2+y^2+1)$ as $(x+y+1)f(x,y)$, where

$$f(x,y)=\left[2x^3(x-y-1)\right]+\left[x(2y^2-y+2)(x-y-1)\right]$$
$$+\left[2y^4-2y^3\right]+\left[2y^2-2y\right]+2.$$

If $y=x$, then $f(x,y)=2x^4-6x^3+3x^2-4x+2$, which has no integer roots by the rational root theorem. Note that x and y cannot both be negative. By symmetry, it suffices to show that $f(x,y)\neq 0$ for $x\geq y+1$ with $x\geq 0$. In this case, observe that each of the bracketed terms in $f(x,y)$ is nonnegative, so that $f(x,y)>0$.

Also solved by Reza Akhlaghi, Roy Barbara (Lebanon), Matt Baker (graduate student), J. C. Binz (Switzerland), Stan Byrd and Terry J. Walters, John Christopher, Con Amore Problem Group (Denmark), Daniele Donini (Italy), David Doster, Arthur H. Foss, Jiro Fukuta (Japan), Marty Getz and Dixon Jones, Thomas Jager, Kee-Wai Lau (China), Atar Sen Mittal, Kandasamy Muthuvel, Oklahoma State University Problem Solving Group, Allan Pedersen (Denmark), Gao Peng (graduate student), John P. Robertson and James S. Robertson, Kenneth Rogers, Nicholas C. Singer, The TAMUK Problem Solvers, Charles H. Webster, Western Maryland College Problems Group, Michael Woltermann, and the proposers. There were eight incorrect solutions and three incomplete solutions.

Answers

Solutions to the Quickies on page 390.

A885. Given $A\in S$, there exist positive integers j and k, with $2j<k$ such that $A^j=A^k$. Let $M=A^{k-j}$. Then

$$M^2=A^{2(k-j)}=A^{k-2j}A^k=A^{k-2j}A^j=A^{k-j}=M.$$

The eigenvalues of M are thus 0 and 1, and Trace(M) is the multiplicity of 1 in the characteristic polynomial of M.

A886. Let $f(a,b,c):=ab^2+bc^2+ca^2$. Then

$$f(a,b,c)+f(a,c,b)=(a+b+c)(ab+bc+ca)-3abc=3(ab+bc+ca-abc)$$
$$=3[(1-a)(1-b)(1-c)+(a+b+c)-1]=3[(1-a)(1-b)(1-c)+2].$$

Because $c\leq 1\leq a$, we have $(1-a)(1-b)(1-c)\leq 0$ if $b\leq 1$. If $1<b$, then

$$(1-a)(1-b)(1-c)\leq\left(\frac{a+b}{2}-1\right)^2(1-c)\leq\left(\frac{3}{2}-1\right)^2(1-0)=\frac{1}{4}.$$

Therefore, $f(a,b,c)+f(a,c,b)\leq 27/4$. Noting that

$$f(a,c,b)-f(a,b,c)=(a-b)(b-c)(a-c)\geq 0,$$

we have $f(a,b,c)\leq 27/8$.

Correction

Q880, June 1998. The first exponent in the sum was incorrect. The problem should have read: Show that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^{(n-k)^n}=n!$.

Acknowledgments. The editors would like to thank Murray S. Klamkin, Loren C. Larson, Harvey Schmidt, and Daniel H. Ullman for their help in reviewing problem proposals over the last year.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Singh, Simon, Mathematics "proves" what the grocer always knew oranges [sic], *New York Times* (25 August 1998) F3. Devlin, Keith, Kepler's sphere packing problem solved, `http://www.maa.org/devlin/devlin_9_98.html` . Peterson, I., Cracking Kepler's sphere-packing problem, *Science News* 154 (15 August 1998) 103. Mackenzie, Dana, The proof is in the packing, *American Scientist* (November–December 1998) `http://www.amsci.org/amsci/issues/Sciobs98/sciobs98-11packing.html` . Hales, Thomas, The Kepler Conjecture, `http:/www.math.lsa.umich.edu/~hales/countdown/` .

Kepler conjectured in 1611 that the densest way to pack spheres in infinite space is the face-centered cubic packing, used by grocers to stack oranges. Thomas Hales (University of Michigan) has announced a proof of the conjecture, which involves substantial computer support and verification. The method involves classifying the possible kinds of star-shaped gaps between spheres, decomposing them into a hybrid of Voronoi cells and Delaunay triangulations, and solving an enormous optimization problem for each kind. The prose part of the proof (250 pp) is at Hales's Web site, as is the 3 GB of programs and data. It will take some time before experts pronounce an opinion on the correctness of the proof!

Morris, S. Brent, *Magic Tricks, Card Shuffling and Dynamic Computer Memories*, MAA, 1998; xv + 148 pp,$28.95 (P). ISBN 0–88385–527–3.

This is a fun book that neatly encapsulates the mathematics behind card shuffling and wraps it beautifully in the milieu of magic. Each chapter begins with the description of a card trick, followed by the development of the mathematics involved, and ends with explaining in terms of the mathematics how the trick works. One chapter shows how card shuffling can be applied to data retrieval and to data interchange in a parallel computer. The level of mathematics involved climbs in the course of the book, from modular arithmetic to permutation groups; but the book can be enjoyed by anyone.

Maor, Eli, *Trigonometric Delights*, Princeton University Press, 1998; xiv + 236 pp, $24.95. ISBN 0–691–05754–0.

Can your students explain why it is advantageous to measure angles in radians? And do you know how recently the term was coined? (1871). True to its title, this book presents delights, both practical and esthetic, that would liven up any student's experience of studying trigonometry. Although the book is not a comprehensive history of trigonometry, it presents numerous topics in trigonometry from a historical perspective, from the Babylonian tablet Plimpton 322 through measuring the earth to Fourier series. Calculus occurs in just a few places (e.g., in considering $(\sin\theta)/\theta$, which arises in calculating the circumference of a circle at latitude $\pi/2-\theta$). [One error marred my enjoyment: The name of A.B. Chace, who investigated the Rhind Papyrus, appears consistently as "Chase."]

Henze, Norbert, and Hans Riedwyl, *How to Win More: Strategies for Increasing a Lottery Win*, A K Peters, 1998; x + 149 pp, $15.95 (P). ISBN 0–56881–078–4.

"The main purpose of *How to Win More* is to give you valuable insights into how to improve your long-term return on investment when playing lotto ... [F]ormulae for computing odds or expectations have been 'banned' into a separate unit (Chapter 8)." The insights boil down to the advice to avoid popular combinations so that you won't be as likely to have to share the prize if you win. The authors define the *arithmetic complexity* of a combination for an r/s lottery as the number of positive differences between the numbers, minus $(r-1)$ (e.g., the arithmetic complexity of an arithmetic progression is 0). Simple rules for generating combinations, which are likely to be used by many people, generally produce combinations of low arithmetic complexity. The Mathematical Appendix is valuable in collecting together in one place a number of formulas (on waiting times, sums of numbers, etc.). The topic of "intelligent play" is treated also in "Lotto play: The good, the fair, and the truly awful," by Dan Kadell and Donald Ylvisaker, *Chance* 4 (1991) (3) 22–25, 57. The book, however, avoids giving advice on *when* to buy a lottery ticket; for that, see "When to buy a lottery ticket," by Sam C. Saunders, *Mathematics Notes from Washington State University* 30 (May 1987) (1–2) (Whole Numbers 117, 118) (but beware inverted exponent in equation (13)).

Dudley, Underwood, *Numerology, or, What Pythagoras Wrought*, MAA, 1997; viii + 316 pp, $29.95. ISBN 0–88385–524–0.

This is a highly entertaining book by a former Associate Editor of this MAGAZINE, whose writing pulls no punches. "Mysticism is a nonrational method of getting at truth ... There is nothing wrong with mysticism. On the other hand, everything is wrong with numerology. Numerologists purport to *apply* number mysticism ... Numerologists assert that numbers tell you where it would be best to live, who you should marry, even at what time you should arrive for an appointment. Numbers do not do this. It is not their job. Numbers have power, but not that kind of power." The chapters of the book tour through the history and current practice of numerology, from Pythagoras ("shame on him"), biblical sevens, rithmomachy (a game), and pyramidology, to the Elliott Wave (explain the stock market with Fibonacci numbers) and biorhythms (good and bad days based on 23, 28, and 33). "[I]t is my hope that copies of [this book] will turn up on the New Age shelves of used book stores, where they may fall into the hands of those expecting something different. The shock may do them good."

Dershowitz, Nachum and Edward M. Reingold, *Calendrical Calculations*, Cambridge University Press, 1997; xxi + 307 pp, $64.95, $22.95 (P). ISBN 0–521–56413–1, 0–521–56474–3.

This book gives precise descriptions of fourteen calendars of current and historical interest, together with accurate algorithms and Lisp computer code. Calendars included are Gregorian, ISO, Julian, Coptic, Ethiopic, Islamic, modern Persian, Bahá'í, Hebrew, Mayan, French Revolutionary, Chinese, old Hindu, and modern Hindu.

Connelly, Robert, and Allen Back, Mathematics and tensegrity, *American Scientist* (March–April 1998) 142–151.

R. Buckminster Fuller popularized tensegrity structures, in which rigid struts are interconnected with cables under tension. This paper describes how these structures can be modeled mathematically and investigates conditions for stability. Group representations come into play, but the authors deftly hold the technical details back in favor of insightful prose exposition.

NEWS AND LETTERS

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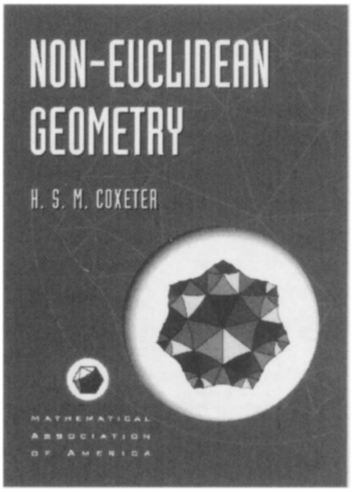

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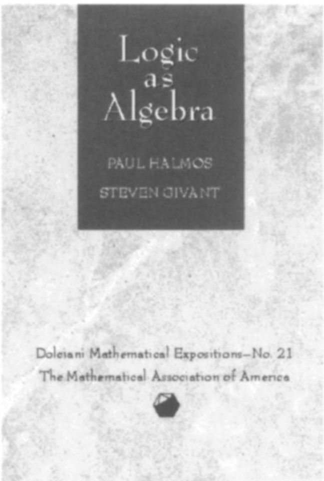

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